On the category of Lie n-algebroids

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Abstract

Lie n-algebroids and Lie infinity algebroids are usually thought of exclusively in supergeometric or algebraic terms. In this work, we apply the higher derived brackets construction to obtain a geometric description of Lie n-algebroids by means of brackets and anchors. Moreover, we provide a geometric description of morphisms of Lie n-algebroids over different bases, give an explicit formula for the Chevalley-Eilenberg differential of a Lie n-algebroid, compare the categories of Lie n-algebroids and NQ-manifolds, and prove some conjectures of Sheng and Zhu [SZ11].

Keywords : Lie n-algebroids, split NQ-manifolds, morphisms, Chevalley-Eilenberg complex, graded symmetric tensor coalgebra, higher derived brackets, Lie infinity (anti)-algebra..

Contents

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1 Introduction

1.1 General background

The starting point of this work is the paper [BKS04] by Bojowald, Kotov, and Strobl. The authors prove that for the Poisson sigma model (PSM) a bundle map is a solution of the field equations if and only if it is a map of Lie algebroids, i.e. a morphism of Q-manifolds, or, as well, of differential graded algebras (DGA). Moreover, gauge equivalent solutions are homotopic maps of Lie algebroids or homotopic morphisms of Q-manifolds.

In case of the AKSZ sigma model, the target space is a symplectic Lie n-algebroid [AKSZ97], see also [Sev01]. The concept of Lie n-algebroid or, more generally, of Lie infinity algebroid can be discussed in the cohesive $(\infty,1)$-topos of synthetic differential ∞-groupoids. Presentations by DGA-s and simplicial presheaves can be given. However, in contrast with Lie algebroids, no interpretation in terms of brackets seems to exist – the approach is essentially algebraic.

Lie infinity algebroids are not only homotopifications of Lie algebroids, but also horizontal categorifications of Lie infinity algebras. Truncated Lie infinity algebras are themselves tightly connected with vertical categorifications of Lie algebras [BC04], [KMP11]. Beyond this categorical approach to Lie infinity algebras, there exists a well-known operadic definition that has the advantage to be valid also for other types of algebras: if P denotes a quadratic Koszul operad, the P_∞-operad is defined as the cobar construction $\Omega P^{¡}$ of the Koszul-dual cooperad $P^{¡}$ of P. A P_∞-structure on a graded vector space V is then a representation on V of the differential graded operad P_∞. On the other hand, a celebrated result by Ginzburg and Kapranov states that P_∞-structures on V are 1:1 (in the finite-dimensional context) with differentials

$$d \in \mathrm{Der}^1(\mathscr{F}^{\mathrm{gr}}_{P^!}(sV^*)) \quad \text{or} \quad d \in \mathrm{Der}^{-1}(\mathscr{F}^{\mathrm{gr}}_{P^!}(s^{-1}V^*)) ,$$

or, also, 1:1 with codifferentials

$$D \in \mathrm{CoDer}^1(\mathscr{F}^{\mathrm{gr,c}}_{P^!}(s^{-1}V)) \quad \text{or} \quad D \in \mathrm{CoDer}^{-1}(\mathscr{F}^{\mathrm{gr,c}}_{P^!}(sV)) .$$

Here $\mathrm{Der}^1(\mathscr{F}^{\mathrm{gr}}_{P^!}(sV^*))$ (resp., $\mathrm{CoDer}^1(\mathscr{F}^{\mathrm{gr,c}}_{P^!}(s^{-1}V))$), for instance, denotes the space of endomorphisms of the free graded algebra over the Koszul dual operad $P^!$ of P on the suspended linear dual sV^* of V, which have degree 1 and are derivations with respect to each binary operation in $P^!$ (resp., the space of endomorphisms of the free graded coalgebra on the desuspended space $s^{-1}V$ that are coderivations) (by differential and codifferential we mean of course a derivation or coderivation that squares to 0). In case P is the Lie operad, this implies that we have a 1:1 correspondence between Lie infinity structures on V and degree 1 differentials $D \in \mathrm{Der}^1(\odot(sV^*))$ of the free graded symmetric algebra over sV^* or degree 1 codifferentials $D \in \mathrm{CoDer}^1(\odot^c(s^{-1}V))$ of the free graded symmetric coalgeba over $s^{-1}V$. The latter description can also be formulated in terms of formal supergeometry (which goes back to Kontsevich's work on Deformation Quantization of Poisson Manifolds): a Lie infinity structure on V is the same concept as a homological vector field on the formal supermanifold $s^{-1}V$.

Each one of the preceding approaches to Lie infinity algebras has its advantages: a number of notions are God-given in the categorical setting, operadic techniques favor conceptual and 'universal' ideas, morphisms tend to live in the algebraic or supergeometric world... However: Lie infinity algebras were originally defined by Lada and Stasheff [LS93] in terms of infinitely many brackets. The same holds true for Lie algebroids: although they are exactly Q-manifolds, i.e. homological vector fields $Q \in \mathrm{Der}^1(\Gamma(\wedge E^*))$ of a split supermanifold $E[1]$, where E is a vector bundle, their original nature is geometric: they are vector bundles with an anchor map and a Lie bracket on sections that verifies the Leibniz rule with respect to the module structure of the space of sections. Moreover, in case of the PSM, Physics is formulated in this geometric setting.

1.2 Main results and structure

In this work, we describe Lie infinity algebroids and their morphisms in terms of anchors and brackets, thus providing a geometric meaning of concepts usually dealt with exclusively in the algebraic or supergeometric frameworks.

Section 2 begins with some information on graded symmetric and graded antisymmetric tensor algebras of $\mathbb{Z}$-graded modules over $\mathbb{Z}$-graded rings. Further, we prove that any N-manifold is non-canonically split, **Theorem 1**, and give examples of canonically split N-manifolds.

In Section 3, we study the standard and the homological gradings of the structure sheaf and the sheaf of vector fields of split N-manifolds, as well as the corresponding Euler fields. We also review graded symmetric tensor coalgebras, their coderivations and cohomomorphisms. We apply the higher derived brackets method to obtain from an NQ-manifold the geometric anchor-bracket description of a Lie n-algebroid, see **Definition 4**. We thus recover and justify a definition given in [SZ11]. There is a 1:1 correspondence between 'geometric' Lie n-algebroids and NQ-manifolds, see **Theorem 2**. The proof leads to the explicit form of the Chevalley-Eilenberg differential of a Lie n-algebroid, see **Definition 5**. It reduces, for $n=1$, to the Lie algebroid de Rham cohomology operator, see **Remark 8**. Moreover, it is shown that any Lie n-algebroid is induced by derived brackets. Some of these results may be considered as natural. However, their proofs are highly complex. Even the concept of 'geometric' Lie n-algebroid was not known before 2011.

The last section contains the definitions of morphisms of 'geometric' Lie n-algebroids over different and over isomorphic bases, see **Definitions 6, 7** and **Remarks 9, 10**. For $n=1$, they reduce to Lie algebroid morphisms [Mac05] and, over a point, we recover Lie infinity algebra morphisms [AP10]. To justify these definitions, we prove that split Lie n-algebroid morphisms are exactly morphisms of NQ-manifolds between split NQ-manifolds, see **Theorem 3**.

2 N-manifolds

2.1 $\mathbb{Z}$-Graded symmetric tensor algebras

The goal of this subsection is to increase readability of our text. The informed reader may skip it. For the $\mathbb{Z}_2$-graded case, see [Man88].

In the following, we denote by $\vee$ (resp., $\wedge$, $\odot$, $\boxdot$) the symmetric (resp., antisymmetric, graded symmetric, graded antisymmetric) tensor product. More precisely, let $M=\oplus_i M_i$ be a $\mathbb{Z}$-graded module over a $\mathbb{Z}$-graded commutative unital ring R. Graded symmetric (resp., graded antisymmetric) tensors on M are defined, exactly as in the nongraded case, as the quotient of the tensor algebra $TM=\oplus_p M^{\otimes p}$ by the ideal

$$I=(m\otimes n-(-1)^{mn}n\otimes m)\quad (\text{resp., } I=(m\otimes n+(-1)^{mn}n\otimes m))\ ,$$

where $m,n\in M$ are homogeneous of degree denoted by m,n as well. The tensor module TM admits the following decomposition:

$$TM=\bigoplus_{p\in\mathbb{N}}\bigoplus_{i_1\leq\ldots\leq i_p} M_{i_1\ldots i_p}:=\bigoplus_{p\in\mathbb{N}}\bigoplus_{i_1\leq\ldots\leq i_p}\left(\bigoplus_{\sigma\in\mathrm{Perm}} M_{\sigma_{i_1}}\otimes\ldots\otimes M_{\sigma_{i_p}}\right)\ ,$$

where Perm denotes the set of all permutations (with repetitions) of the elements $i_1 \leq \ldots \leq i_p$. For instance, if $M = M_0 \oplus M_1 \oplus M_2$, the module $M^{\otimes 3}$ is given by

$$M^{\otimes 3} = M_0 \otimes M_0 \otimes M_0 \oplus (M_0 \otimes M_0 \otimes M_1 \oplus M_0 \otimes M_1 \otimes M_0 \oplus M_1 \otimes M_0 \otimes M_0) \oplus \ldots$$

The ideal I is homogeneous, i.e.

$$I = \bigoplus_{p \in \mathbb{N}} \bigoplus_{i_1 \leq \ldots \leq i_p} \left(M_{i_1 \ldots i_p} \cap I\right) ,$$

which actually means that the components of the decomposition of any element of $I \subset TM$ are elements of I as well. To check homogeneity, it suffices to note that an element of I, e.g.

$$(m_0 \otimes m_1 - m_1 \otimes m_0) \otimes (m_0' + m_1') ,$$

where the subscripts denote the degrees, has components

$$(m_0 \otimes m_1 \otimes m_0' - m_1 \otimes m_0 \otimes m_0') + (m_0 \otimes m_1 \otimes m_1' - m_1 \otimes m_0 \otimes m_1')$$

in I, since we accept permutations inside the components. It follows that the graded symmetric tensor algebra $\odot M = TM/I$ reads

$$\odot M = \bigoplus_{p \in \mathbb{N}} \bigoplus_{i_1 \leq \ldots \leq i_p} M_{i_1} \odot \ldots \odot M_{i_p}, \quad \text{where} \quad M_{i_1} \odot \ldots \odot M_{i_p} := M_{i_1 \ldots i_p} / (M_{i_1 \ldots i_p} \cap I) .$$

The 'same' result holds true in the graded antisymmetric situation.

Note that, if R is a $\mathbb{Q}$-algebra, the module $M_0 \odot M_1$ is isomorphic to $M_0 \otimes M_1$, via

$$[(1/2)(m_0 \otimes m_1 + m_1 \otimes m_0)] \mapsto m_0 \otimes m_1 .$$

As the tensors $m_0 \otimes m_1$ and $\frac{1}{2}(m_0 \otimes m_1 + m_1 \otimes m_0)$ differ by $\frac{1}{2}(m_0 \otimes m_1 - m_1 \otimes m_0)$ and thus coincide in the quotient $M_0 \odot M_1$, the identification $M_0 \odot M_1 \simeq M_0 \otimes M_1$ corresponds to the choice of a specific representative. Moreover, we have module isomorphisms of the type

$$M_0 \odot M_0 \odot M_0 \simeq \vee^3 M_0 , \quad M_0 \odot M_0 \odot M_1 \simeq \vee^2 M_0 \otimes M_1 , \quad M_0 \odot M_1 \odot M_1 \simeq M_0 \otimes \wedge^2 M_1 .$$

Similarly,

$$M_0 \boxdot M_0 \boxdot M_0 \simeq \wedge^3 M_0 , \quad M_0 \boxdot M_0 \boxdot M_1 \simeq \wedge^2 M_0 \otimes M_1 , \quad M_0 \boxdot M_1 \boxdot M_1 \simeq M_0 \otimes \vee^2 M_1 .$$

2.2 Batchelor's theorem for N-manifolds

The results of this section are well-known. We consider their proofs as a warm-up exercise (although we did not find them in the literature).

Definition 1. *An* N-manifold *or* $\mathbb{N}$-graded manifold *of degree n (and dimension $p|q_1|\ldots|q_n$) is a smooth Hausdorff second-countable (hence paracompact) manifold M endowed with a sheaf $\mathscr{A}$ of $\mathbb{N}$-graded commutative associative unital $\mathbb{R}$-algebras, whose degree 0 term is $\mathscr{A}^0 = C^\infty_M$ and which is locally freely generated, over the corresponding sections of C^∞_M (i.e. over functions of p variables x^j of degree 0), by $q_1, \ldots, q_n$ graded commutative generators $\xi_1^{j_1}, \ldots, \xi_n^{j_n}$ of degree $1, \ldots, n$, respectively.*

If necessary, we assume – for simplicity – that M is connected. Moreover, it is clear that coordinate transformations are required to preserve the $\mathbb{N}$-degree. An alternative definition of N-manifolds by coordinate charts and transition maps can be given.

Example 1. Just as a vector bundle with shifted parity in the fibers is a (split) supermanifold, a graded vector bundle with shifted degrees is a (split) N-manifold. More precisely, let $E_{-1}, E_{-2}, \ldots, E_{-n}$ be smooth vector bundles of finite rank $q_1, \ldots, q_n$ over a same smooth (Hausdorff, second countable) manifold M. If we assign the degree i to the fiber coordinates of E_{-i}, we get an N-manifold $E_{-i}[i]$. The direct sum $E = \oplus_{i=1}^n E_{-i}$ (resp., $E[\cdot] = \oplus_{i=1}^n E_{-i}[i]$) is a graded vector bundle concentrated in degrees -1 to $-n$, with local coordinates of degree 0 (resp., with local fiber coordinates of degrees $1, \ldots, n$. We denote the graded symmetric tensor algebra of the dual bundle $E^* = \oplus_{i=1}^n E^*_{-i}$, where the vectors of E^*_{-i} have degree i, by $\odot E^*$. Consider first the case $n = 2$. We have

$$\odot E^* = \odot(E^*_{-1} \oplus E^*_{-2}) = \odot E^*_{-1} \otimes \odot E^*_{-2} = \wedge E^*_{-1} \otimes \vee E^*_{-2} \; .$$

Hence,

$$\mathscr{A} := \Gamma(\odot E^*) = \Gamma(\wedge E^*_{-1} \otimes \vee E^*_{-2}) = \oplus_{n \in \mathbb{N}} \oplus_{k+2\ell=n} \Gamma(\wedge^k E^*_{-1} \otimes \vee^\ell E^*_{-2}) \; .$$

In particular,

$$\mathscr{A}^0 = C^\infty_M, \; \mathscr{A}^1 = \Gamma(E^*_{-1}), \; \mathscr{A}^2 = \Gamma(\wedge^2 E^*_{-1} \oplus E^*_{-2}), \; \ldots$$

Note that, if we work over a common local trivialization domain $U \subset M$ of E_{-1} and E_{-2}, and denote the base coordinates by x and the fiber coordinates of E_{-1} and E_{-2} by ξ and η, respectively (ξ and η are then also the base vectors of the fibers of E^*_{-1} and E^*_{-2}), a function f in $\mathscr{A}^2(U)$ reads

$$f(x, \xi, \eta) = \sum_{i<j} f_{ij}(x) \xi^i \xi^j + \sum_a f_a(x) \eta^a \; .$$

Similar results hold of course for any n. Eventually, the sheaf $(M, \mathscr{A})$ is an N-manifold of degree n. It is clear that this manifold should be denoted by $E[\cdot] = \oplus_{i=1}^n E_{-i}[i]$.

Definition 2. *A N-manifold $E[\cdot] = \oplus_{i=1}^n E_{-i}[i]$ induced by a graded vector bundle is called a* split N-manifold*.*

Batchelor's theorem [Bat80] states that in the smooth category any supermanifold is noncanonically diffeomorphic to a split supermanifold. A similar result is known to hold true for N-manifolds.

Theorem 1. *Any N-manifold $(M, \mathscr{A})$ of degree n is noncanonically diffeomorphic to a split N-manifold $E[\cdot]$, where $E = \oplus_{i=1}^n E_{-i}$ is a graded vector bundle over M concentrated in degrees $-1, \ldots, -n$.*

Sketch of Proof. Consider first an N-manifold $(M, \mathscr{A})$ of degree $n = 2$. Since $\mathscr{A}^0 = C^\infty_M$ and $\mathscr{A}^0 \mathscr{A}^1 \subset \mathscr{A}^1$, the sheaf $\mathscr{A}^1$ is a locally free sheaf of C^∞_M-modules and

$$\mathscr{A}^1 \simeq \Gamma(E^*_{-1}),$$

for some vector bundle $E_{-1} \to M$. Let now $\mathscr{A}_1$ be the subalgebra of $\mathscr{A}$ generated by $\mathscr{A}^0 \oplus \mathscr{A}^1$. Clearly,

$$\mathscr{A}_1 = \mathscr{A}^0 \oplus \mathscr{A}^1 \oplus (\mathscr{A}^1)^2 \oplus \ldots \simeq \Gamma(\wedge E^*_{-1}) \tag{1}$$

and $\mathscr{A}_1 \cap \mathscr{A}^2 = (\mathscr{A}^1)^2$ is a proper $\mathscr{A}^0$-submodule of $\mathscr{A}^2$. Since the quotient $\mathscr{A}^2/(\mathscr{A}^1)^2$ is a locally free sheaf of C^∞_M-modules, we have

$$\mathscr{A}^2/(\mathscr{A}^1)^2 \simeq \Gamma(E^*_{-2}),$$

where E_{-2} is a vector bundle over M. The short exact sequence

$$0 \to (\mathscr{A}^1)^2 \to \mathscr{A}^2 \to \Gamma(E^*_{-2}) \to 0 \tag{2}$$

of $\mathscr{A}^0$-modules is non canonically split. Indeed, the Serre-Swan theorem states that, if N is a smooth (Hausdorff, second-countable) manifold, a $C^\infty(N)$-module is finitely generated and projective if and only if it is the module of smooth sections of a smooth vector bundle over N, see e.g. [Nes03][Theo.11.32], [GMS05]. The aforementioned splitting of the sequence (2) is a direct consequence of the fact that $\Gamma(E^*_{-2})$ is projective. Let us *fix a splitting* and identify $\Gamma(E^*_{-2})$ with a submodule of $\mathscr{A}^2$:

$$\mathscr{A}^2 = (\mathscr{A}^1)^2 \oplus \Gamma(E^*_{-2}) = \Gamma(\wedge^2 E^*_{-1} \oplus E^*_{-2}) .$$

It follows that the subalgebra $\mathscr{A}_2$, which is generated by $\mathscr{A}^0 \oplus \mathscr{A}^1 \oplus \mathscr{A}^2$, reads

$$\mathscr{A}_2 = \Gamma(\wedge E^*_{-1} \otimes \vee E^*_{-2}) .$$

In the case $n = 2$, the algebra $\mathscr{A}_2$ is of course the whole function algebra $\mathscr{A}$. For higher n, we iterate the preceding approach and obtain finally, modulo choices of splittings, that

$$\mathscr{A} = \Gamma(\wedge E^*_{-1} \otimes \vee E^*_{-2} \otimes \wedge E^*_{-3} \otimes \ldots) = \Gamma(\odot E^*) .$$

Hence, the considered N-manifold is diffeomorphic, modulo the chosen splittings, to the split N-manifold $E[\cdot] = \oplus_{i=1}^n E_{-i}[i]$. □

Remarks 1.

- The preceding theorem means that N-manifolds of degree n together with a choice of splittings are 1:1 with graded vector bundles concentrated in degrees $-1, \ldots, -n$.
- In view of Equation (1), an N-manifold of degree 1 is canonically diffeomorphic to a split N-manifold of degree 1.
- Without a choice of splittings, an N-manifold $M = (M_0, \mathscr{A})$ of degree n gives rise to a filtration

  $$\mathscr{A}_0 \subset \mathscr{A}_1 \subset \mathscr{A}_2 \subset \ldots \subset \mathscr{A}_n = \mathscr{A} ,$$

  which implements a tower of fibrations

  $$M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \ldots \leftarrow M_n = M ,$$

  see [Roy02]. Here, $M_1 = E_{-1}[1]$, where $E_{-1} \to M_0$ is a smooth vector bundle, whereas $M_i \to M_{i-1}$, $i > 1$, is only a fibration. Each M_i is an N-manifold of degree i.

Example 2. In case of the N-manifold $M = T^*[2]T[1]M_0$ associated to the standard Courant algebroid, we have the tower

$$M_0 \leftarrow T[1]M_0 \leftarrow T^*[2]T[1]M_0 .$$

Clearly, the N-manifold $M = M_2$ is not canonically of the type $E_{-1}[1] \oplus E_{-2}[2]$: it is only noncanonically split. On the other hand, it is even diffeomorphic, as NQ-manifold, to a split NQ-manifold, see [SZ11][Theorem 3.4].

2.3 Examples of canonically split N-manifolds

Split N-manifolds appear naturally in connection with the integration problem of exact Courant algebroids.

Let us be more precise. A representation of a Lie algebroid (A, ℓ_2, ρ) on a vector bundle E over the same base, say M, is defined very naturally as a Lie algebroid morphism $\delta : A \to \mathscr{A}(E)$, where $\mathscr{A}(E)$ is the Atiyah algebroid associated to E [CM08]. In other words, δ is a $C^\infty(M)$-linear map from $\Gamma(A) \to \Gamma(\mathscr{A}(E)) \simeq \mathrm{Der}(\Gamma(E))$ (where the latter module is the module of derivative endomorphisms of $\Gamma(E)$), which respects the anchors and the Lie brackets. When viewing δ as a map $\delta : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$, we can interpret it as an A-connection on E with vanishing curvature (the anchor condition is automatically verified). It is well-known that a TM-connection on E allows to extend the de Rham operator to E-valued differential forms $\Gamma(\wedge T^*M \otimes E)$. This generalization verifies the Leibniz rule with respect to the tensor product and it squares to 0 if and only if the connection is flat. Similarly, the flat A-connection δ on E, i.e. the representation δ on E of the Lie algebroid A, can be seen as a degree 1 operator d^δ on E-valued bundle forms

$$\Gamma(\wedge A^* \otimes E) ,$$

which is a derivation and squares to 0, i.e as a kind of homological vector field.

Many concepts – even involved ones – are very natural in this supergeometric setting. For instance, to obtain the notion of representation up to homotopy of a Lie algebroid, it suffices to replace in the latter algebraic definition of an ordinary Lie algebroid representation, the vector bundle E by a graded vector bundle E and to ask that the square zero derivation, say D, be of total degree 1. Representations up to homotopy, we denote them by (E, D), provide the appropriate framework for the definition of the adjoint representation of a Lie algebroid and the interpretation of the Lie algebroid deformation cohomology [CM08] as a cohomology associated to a representation. For more details we refer the reader to [AC09]. When retranslated into the original geometric context (explicit transfers of this type can be found below), the preceding supergeometric definition means roughly that a representation up to homotopy of a Lie algebroid A, is a complex (E, ∂) of vector bundles endowed with an A-connection δ that is flat only up to homotopy: $R^\delta = -\partial\omega_2$, the homotopy ω_2 verifies a coherence law $d^\delta\omega_2 = -\partial\omega_3$ up to homotopy ω_3, and so on.

To integrate an exact Courant algebroid [SZ11], Sheng and Zhu view this algebroid as an extension of the tangent bundle by its coadjoint representation up to homotopy and integrate that extension. The extensions of Lie algebroids A by representations up to homotopy (E, D), they consider, are twisted semidirect products of A by (E, D). More precisely, they are twists by cocycles of the representation up to homotopy induced by D on $\odot sE^*$, where s is the suspension operator and E a graded vector bundle. Both, semidirect products and extensions, are canonically split N- and even NQ-manifolds. For instance, a semidirect product is a representation on $\odot sE^*$, i.e. a degree 1 square 0 derivation on

$$\Gamma(\wedge A^* \otimes \odot sE^*) = \Gamma(\odot sA^* \otimes \odot sE^*) = \Gamma(\odot(s^{-1}(A \oplus E))^*) ,$$

and therefore a split NQ-manifold.

3 Geometry of Lie n-algebroids

3.1 Standard and homological gradings of split N-manifolds

Let $\mathscr{M} = (M, \mathscr{A})$ be an N-manifold of degree n, consider local coordinates in an open subset $U \subset M$, and denote by $u = (u^\alpha) = (\xi_1^{j_1}, \ldots, \xi_n^{j_n})$ the coordinates of nonzero degree. The weighted

Euler vector field

$$\varepsilon_U = \sum_\alpha \varpi(u^\alpha) u^\alpha \partial_{u^\alpha} \ ,$$

where $\varpi(u^\alpha)$ denotes the degree of u^α, is well-defined globally: $\varepsilon_U = \varepsilon|_U$, where ε is the degree-derivation of the graded algebra $\mathscr{A}^\bullet = \oplus_k \mathscr{A}^k$, i.e. for any $f \in \mathscr{A}^k$, $\varepsilon f = kf \in \mathscr{A}^k$. Denote now by $\mathrm{Der}^\bullet \mathscr{A}$ the sheaf of graded derivations of $\mathscr{A}$, which is in particular a sheaf of graded Lie algebras – we denote their brackets by $[-,-]$. It is well-known that the grading of these vector fields is captured by the degree zero interior derivation $[\varepsilon,-]$ of $\mathrm{Der}^\bullet \mathscr{A}$: if $X \in \mathrm{Der}^k \mathscr{A}$, then $[\varepsilon, X] = kX \in \mathrm{Der}^k \mathscr{A}$. The gradings of other geometric objects are encrypted similarly, see [Roy02]. On the other hand, it is clear that the standard Euler vector field $\tilde{\varepsilon}_U = \sum_\alpha u^\alpha \partial_{u^\alpha}$, which encodes the local grading by the number of generators, is not global.

Example 4. Consider for instance $\mathscr{M} = T^*[2]T[1]M$, with coordinates $q^j, \xi^j, p_j, \theta_j$, where ξ^j is thought of as dq^j, p_j as ∂_{q^j}, and θ_j as ∂_{ξ^j}, so that they are of degree $0,1,2,1$, respectively. A simple example, e.g. the coordinate transformation $q = Q, \xi = Q\Xi + e^Q \Theta, p = Q\Xi\Theta + P, \theta = Q^2\Xi$, for $\dim M = 1$, allows to check the preceding claim by direct computation.

The situation is different for a split N-manifold $E[\cdot] = \oplus_{i=1}^n E_{-i}[i]$ over M. Its structure sheaf

$$\begin{aligned} \mathscr{A} &= \Gamma(\odot E^*) \\ &= C^\infty_M \oplus \Gamma(E^*_{-1}) \oplus \left(\wedge^2_{C^\infty_M} \Gamma(E^*_{-1}) \oplus \Gamma(E^*_{-2})\right) \\ &\oplus \left(\wedge^3_{C^\infty_M} \Gamma(E^*_{-1}) \oplus \Gamma(E^*_{-1}) \otimes_{C^\infty_M} \Gamma(E^*_{-2}) \oplus \Gamma(E^*_{-3})\right) \oplus \ldots \end{aligned}$$

clearly carries two gradings, the standard grading induced by E (encoded by ε) and the grading by the number of generators (encoded by $\tilde{\varepsilon}$). We refer to the first (resp., second) degree as the *standard* (resp., *homological*) degree and write

$$\mathscr{A}^\bullet = \oplus_k \mathscr{A}^k \quad (\text{resp.,} \quad {}^\bullet\mathscr{A} = \oplus_r {}^r\mathscr{A}) \ .$$

Both Euler fields, ε and $\tilde{\varepsilon}$, are now global: $\tilde{\varepsilon}$ is the degree-derivation of the graded algebra ${}^\bullet\mathscr{A} = \oplus_r {}^r\mathscr{A}$, i.e. its eigenvectors with eigenvalue r are the elements of ${}^r\mathscr{A}$.

Remark 2. Split supermanifolds are not a full subcategory of supermanifolds: their morphisms respect the additional grading in the structure sheaf. For N-manifolds the situation is similar. N-manifolds are nonlinear generalizations of vector bundles [Vor10], in the sense that a coordinate transformation may contain nonlinear terms. If we denote, e.g. in degree $n = 2$, the transformation by $(x^i, \xi^a, \eta^\mathfrak{a}) \leftrightarrow (y^i, \theta^a, \tau^\mathfrak{a})$, this means that the general form of $\tau^\mathfrak{b}$ is

$$\tau^\mathfrak{b} = \tau^\mathfrak{b}_{ab}(x) \xi^a \xi^b + \tau^\mathfrak{b}_\mathfrak{a}(x) \eta^\mathfrak{a} \ .$$

In the case of split N-manifolds, morphisms respect not only the standard degree, but also the homological one:

$$y^j = y^j(x),\ \theta^b = \theta^b_a(x) \xi^a,\ \tau^\mathfrak{b} = \tau^\mathfrak{b}_\mathfrak{a}(x) \eta^\mathfrak{a}$$

and

$$x^i = x^i(y),\ \xi^a = \xi^a_b(y) \theta^b,\ \eta^\mathfrak{a} = \eta^\mathfrak{a}_\mathfrak{b}(y) \tau^\mathfrak{b} \ . \tag{3}$$

Invariance of ε_U and $\tilde{\varepsilon}_U$ is now also easily checked by direct computation.

Due to the additional grading in the structure sheaf $\mathscr{A}$, vector fields of a split N-manifold $(M, \mathscr{A})$, i.e. sections of the sheaf $\mathrm{Der}^\bullet \mathscr{A}$ of graded derivations of $\mathscr{A}$, acquire a homological degree as well:

we say that $X \in \mathrm{Der}^\ell \mathscr{A}$ has homological degree s and we write $X \in {}^s\mathrm{Der}^\ell \mathscr{A}$, if $X({}^r\mathscr{A}) \subset {}^{r+s}\mathscr{A}$. This condition is equivalent to the requirement that $[\tilde{\varepsilon}, X] = sX$. Indeed, $\tilde{\varepsilon} \in \mathrm{Der}^0 \mathscr{A}$ and if $f \in {}^r\mathscr{A}$, then

$$[\tilde{\varepsilon}, X]f = \tilde{\varepsilon}Xf - X\tilde{\varepsilon}f = \tilde{\varepsilon}Xf - rXf \ .$$

Hence, the announced equivalence.

Proposition 1. *The sheaf of vector fields of a split N-manifold $(M, \mathscr{A})$ of degree n is bigraded, i.e.*

$$\mathrm{Der}^\bullet \mathscr{A} = \oplus_{\ell \geq -n} \mathrm{Der}^\ell \mathscr{A} \quad \textit{and} \quad \mathrm{Der}^\ell \mathscr{A} = \oplus_{s \geq -1} {}^s\mathrm{Der}^\ell \mathscr{A} \ .$$

Proof. The first part of the claim is obvious. Let now $X \in \mathrm{Der}^\ell \mathscr{A}$, consider – to simplify notations – the case $n = 2$, and denote, as above, local coordinates in $U \subset M$ by $v = (x, \xi, \eta)$ and in $V \subset M$ by $w = (y, \theta, \tau)$, where superscripts are understood. The vector field X locally reads

$$X|_U = \sum_s \left[f_s(v)\partial_x + g_{s+1}(v)\partial_\xi + h_{s+1}(v)\partial_\eta \right] \tag{4}$$

and

$$X|_V = \sum_s \left[F_s(w)\partial_y + G_{s+1}(w)\partial_\theta + H_{s+1}(w)\partial_\tau \right] \ , \tag{5}$$

where the sum over s refers to the homological grading $\mathscr{A} = \oplus_s {}^s\mathscr{A}$ and where the degree of ∂_x (resp., ∂_ξ, ∂_η) is 0 (resp., -1, -1) with respect to the homological degree (and similarly for $\partial_y, \partial_\theta, \partial_\tau$). In view of (3), we get

$$\partial_x = \partial_x y|_{x=x(y)}\partial_y + \partial_x \theta|_{x=x(y)}\xi(y)\theta\partial_\theta + \partial_x \tau|_{x=x(y)}\eta(y)\tau\partial_\tau =: A(y)\partial_y + B(y)\theta\partial_\theta + C(y)\tau\partial_\tau \ ,$$

$$\partial_\xi =: D(y)\partial_\theta \quad \text{and} \quad \partial_\eta =: E(y)\partial_\tau \ .$$

Hence, on $U \cap V$,

$$X|_U = \sum_s \left[f_s(w)A(y)\partial_y + (f_s(w)B(y)\theta + g_{s+1}(w)D(y))\partial_\theta \right.$$

$$\left. + (f_s(w)C(y)\tau + h_{s+1}(w)E(y))\partial_\tau \right] \ . \tag{6}$$

It follows that the coefficients of $\partial_y, \partial_\theta, \partial_\tau$ and therefore the brackets in (5) and (6) coincide on $U \cap V$. This means that the brackets in (4) and (5), which are elements of ${}^s\mathrm{Der}^\ell \mathscr{A}(U)$ and ${}^s\mathrm{Der}^\ell \mathscr{A}(V)$, respectively, are the restrictions of a global vector field $X^s \in {}^s\mathrm{Der}^\ell \mathscr{A}$. Eventually, $X = \sum_s X^s$ and this decomposition is obviously unique. □

3.2 $\mathbb{Z}$-Graded symmetric tensor coalgebras

In the following, we will need some results on the graded symmetric tensor coalgebra.

If V is a graded vector space, we denote by $\bar{\odot} V = \oplus_{r \geq 1} \odot^r V$ the reduced graded symmetric tensor algebra over V. If W denotes a graded vector space as well, any linear map $f \in \mathrm{Hom}(\bar{\odot}V, \bar{\odot}W)$ is defined by its ‘restrictions’ $f_{rs} \in \mathrm{Hom}(\odot^r V, \odot^s W)$, $r, s \geq 1$. The graded symmetric product

$$f_{r_1 s_1} \odot g_{r_2 s_2} \in \mathrm{Hom}(\odot^{r_1+r_2} V, \odot^{s_1+s_2} W)$$

is defined, for a homogeneous $g_{r_2 s_2}$ – of degree g with respect to the standard grading –, by

$$(f_{r_1 s_1} \odot g_{r_2 s_2})(v_1, \ldots, v_{r_1+r_2}) =$$

$$\sum_{\sigma\in \mathrm{Sh(r_1,r_2)}} (-1)^{g(v_{\sigma_1}+\ldots+v_{\sigma_{r_1}})}\varepsilon(\sigma)\, f_{r_1s_1}(v_{\sigma_1},\ldots,v_{\sigma_{r_1}})\odot g_{r_2s_2}(v_{\sigma_{r_1+1}},\ldots,v_{\sigma_{r_1+r_2}})\;, \tag{7}$$

where the v_k are homogeneous elements of V of degree denoted by v_k as well, and where $\varepsilon(\sigma)$ is the Koszul sign. If f,g are homogeneous, we have of course

$$g\odot f = (-1)^{fg} f\odot g\;.$$

The graded symmetric tensor product of linear maps is needed to construct coderivations and cohomomorphisms of graded symmetric tensor coalgebras from their corestrictions. The reduced graded symmetric coalgebra on a graded vector space V is made up by the tensor space $\bar{\odot}V=\oplus_{r\geq 1}\odot^r V$ endowed with the coproduct

$$\Delta(v_1\odot\ldots\odot v_r):=\sum_{k=1}^{r-1}\sum_{\sigma\in \mathrm{Sh}(k,r-k)}\varepsilon(\sigma)(v_{\sigma_1}\odot\ldots\odot v_{\sigma_k})\otimes(v_{\sigma_{k+1}}\odot\ldots\odot v_{\sigma_r})\;,$$

with self-explaining notations. We denote the symmetric coalgebra on V by $\bar{\odot}^cV$.

A coderivation $\delta:\bar{\odot}^cV\to\bar{\odot}^cV$ (resp., a cohomomorphism $\phi:\bar{\odot}^cV\to\bar{\odot}^cW$) is completely defined by its corestrictions $\delta_r:\odot^rV\to V$ (resp., $\phi_r:\odot^rV\to W$), $r\geq 1$:

$$\delta(v_1\odot\ldots\odot v_r)=\sum_{k=1}^{r}(\delta_k\odot \mathrm{id}_{r-k})(v_1\odot\ldots\odot v_r)\;, \tag{8}$$

(resp.,

$$\phi(v_1\odot\ldots\odot v_r)=\sum_{s=1}^{r}\frac{1}{s!}\sum_{\substack{r_1+\ldots+r_s=r\\ r_i\neq 0}}(\phi_{r_1}\odot\ldots\odot\phi_{r_s})(v_1\odot\ldots\odot v_r)\;.) \tag{9}$$

These results are just a matter of computation.

The next facts about the tensor power of the suspension operator will be useful. If $E=\oplus_{i=1}^{n}E_{-i}$ is as usually a graded vector bundle and if $s:\Gamma(E_{-k})\to\Gamma((sE)_{-k+1})$ denotes the shift operator, the assignment

$$\Gamma(E_{-k})^{\times 2}\ni(X_1,X_2)\mapsto(-1)^{X_1}sX_1\boxdot sX_2\in\boxdot^2_{C^\infty_M}\Gamma((sE)_{-k+1})$$

is graded symmetric and C^∞_M-bilinear, and thus defines a C^∞_M-linear map on $\odot^2_{C^\infty_M}\Gamma\,(E_{-k})$. More generally, for $a_1\leq\ldots\leq a_i$, set

$$s^i:\Gamma(E_{-a_1})\odot_{(C^\infty_M)}\ldots\odot_{(C^\infty_M)}\Gamma(E_{-a_i})\ni X_1\odot\ldots\odot X_i\mapsto$$

$$(-1)^{\sum_j(i-j)X_j}sX_1\boxdot\ldots\boxdot sX_i\in\Gamma((sE)_{-a_1+1})\boxdot_{(C^\infty_M)}\ldots\boxdot_{(C^\infty_M)}\Gamma((sE)_{-a_i+1})\;,$$

where the tensor products in the source and target spaces are taken, either over C^∞_M, or over $\mathbb{R}$. The inverse of s^i is given by

$$(s^i)^{-1}=(-1)^{i(i-1)/2}(s^{-1})^i\;,$$

where s^{-1} is the desuspension operator.

Just as for Lie infinity algebras, we will find two variants of the definition of Lie infinity algebroids. It is important to first fully understand the purely algebraic situation. Let $\ell'\in\mathrm{Codiff}^1(\bar{\odot}^cV)$ be a degree 1 codifferential of the reduced graded symmetric tensor coalgebra of a desuspended graded vector space $V = s^{-1}W$. The condition $\ell'^2 = 0$ is satisfied if and only if the projection pr_1 onto V of the restriction to $\odot^r V$ of ℓ'^2 vanishes for all $r \geq 1$, i.e. if, see (8),

$$\mathrm{pr}_1 \sum_{i=1}^{r} \sum_{j=1}^{r-i+1} (\ell'_j \odot \mathrm{id}_{r-i-j+1})(\ell'_i \odot \mathrm{id}_{r-i}) = \sum_{i+j=r+1} \ell'_j(\ell'_i \odot \mathrm{id}_{r-i}) = 0, \text{ for all } r \geq 1 \ ,$$

where $\ell'_i : \odot^i V \to V$ is the i-th corestriction of ℓ' (it is of course of degree 1). This structure was discovered by Voronov [Vor05] via derived brackets under the name of L_∞-antialgebra.

Definition 3. *An* L_∞-antialgebra *structure on a graded vector space V is made up by graded symmetric multilinear maps $\ell'_i : V^{\times i} \to V$, $i \geq 1$, of degree 1, which verify the conditions*

$$\sum_{i+j=r+1} \sum_{\sigma \in \mathrm{Sh}(i,j-1)} \varepsilon(\sigma) \ell'_j(\ell'_i(v_{\sigma_1}, \ldots, v_{\sigma_i}), v_{\sigma_{i+1}}, \ldots, v_{\sigma_r}) = 0 \ , \tag{10}$$

for all homogeneous $v_k \in V$ and all $r \geq 1$.

The definition of an L_∞-algebra is similar, except that the multilinear maps are of degree $2-i$ and graded antisymmetric, and that the sign $\varepsilon(\sigma)$ in (10) is replaced by $(-1)^{i(j-1)}\,\mathrm{sign}(\sigma)\varepsilon(\sigma)$, where $\mathrm{sign}(\sigma)$ denotes the signature. See [LS93] and Definition 4, Equation (14) of the present text.

Proposition 2. *An L_∞-antialgebra structure ℓ'_i, $i \geq 1$, on $V = s^{-1}W$ induces an L_∞-algebra structure $\ell_i := s\,\ell'_i\,(s^{-1})^i$ on W, and, vice versa, an L_∞-algebra structure ℓ_i, $i \geq 1$, on W implements an L_∞-antialgebra structure $\ell'_i := (-1)^{i(i-1)/2} s^{-1}\,\ell_i\,s^i$ on V.*

The result is just a reformulation of the construction of a Lie infinity algebra from a codifferential. Let us nevertheless emphasize that the sign in the preceding correspondence is important (and can for instance not be transferred from the definition of ℓ'_i to that of ℓ_i).

3.3 Higher derived brackets construction of Lie n-algebroids

We already mentioned that Lie algebroids are 1:1 with Q-manifolds. In this section we extend this correspondence to Lie n-algebroids and NQ-manifolds of degree n. Many authors consider Lie n-algebroids, but actually just mean NQ-manifolds [Sev01]: it is only in 2011 that Sheng and Zhu defined split Lie n-algebroids by means of anchors and brackets [SZ11]. They mention the bijection with split NQ-manifolds, but give no proof. It turned out that the latter is highly technical. Below, we provide two possible approaches to this correspondence. In particular, we prove that the orders of the brackets (as differential operators) of a Lie n-algebroid suggested in [SZ11] are the only possible ones. For algebroids with generalized anchors, see [GKP11].

Remark 3. Many concepts of Lie n-algebras appear in the literature. Lie n-algebras are in principle specific linear $(n-1)$-categories. But the term 'Lie n-algebras' often also refers to n-ary Lie algebras and to n-term Lie infinity algebras. However, whereas Lie 2-algebras in the categorical sense and 2-term Lie infinity algebras are the objects of two 2-equivalent 2-categories [BC04], 'categorical' Lie 3-algebras are (in 1:1 correspondence with) quite particular 3-term Lie infinity algebras (their bilinear and trilinear maps have to vanish in degree (1,1) and in total degree 1, respectively). The main reason for these complications is that the map

$$\boxtimes : L \times L' \mapsto L \boxtimes L' \ ,$$

where $\boxtimes$ denotes the monoidal structure of the category `Vect` n-`Cat` of linear n-categories, is not a bilinear n-functor [KMP11]. Nevertheless, when speaking about a Lie n-algebra (resp., algebroid), we mean in this text an n-term Lie infinity algebra (resp., algebroid).

One of the possible approaches to split Lie n-algebroids is Voronov's higher derived brackets construction [Vor05], which we now briefly recall. Let L be a Lie superalgebra with bracket $[-,-]$ and let $P \in \mathrm{End}\, L$ be a projector, such that $V = P(L)$ be an abelian Lie subalgebra and

$$P[\ell, \ell'] = P[P\ell, \ell'] + P[\ell, P\ell'] ,$$

for any $\ell, \ell' \in L$. The latter condition is just a convenient way to say that $\mathrm{Ker}\, P$ is a Lie subalgebra as well. Let us mention that this setup implies that $L = V \oplus K$, $K = \mathrm{Ker}\, P$. Consider now an odd derivation $D \in \mathrm{Der}\, L$ that respects K, i.e. $DK \subset K$, and construct higher derived brackets on V:

$$\{v_1, \ldots, v_k\}_D := P[\ldots[[Dv_1, v_2], v_3], \ldots, v_k] .$$

If $D^2(V) = 0$, this sequence of k-ary brackets, $k \geq 1$, defines a Lie infinity antialgebra structure on V and induces a Lie infinity algebra structure on sV.

Next we consider a split N-manifold $E[\cdot] = \oplus_{i=1}^n E_{-i}[i]$ of degree n over a base manifold M. The interior product of an element of $\mathscr{A} = \Gamma(\odot E^*) = \odot_{C^\infty(M)} \Gamma(E^*)$ by $X_j \in \Gamma(E_{-j})$ is defined by 0 on $f \in C^\infty(M)$, on the other generators $\omega_k \in \Gamma(E^*_{-k})$ by

$$i_{X_j} \omega_k = \delta_{jk} (-1)^{jk} \omega_k(X_j) \in \mathscr{A}^{k-j} ,$$

where δ_{jk} is Kronecker's symbol, and it is extended to the whole graded symmetric algebra $\mathscr{A}^\bullet$ as a derivation of degree $-j$, i.e. by

$$i_{X_j}(S \odot T) = (i_{X_j} S) \odot T + (-1)^{j\ell} S \odot (i_{X_j} T) ,$$

where $S \in \mathscr{A}^\ell$ and $T \in \mathscr{A}$. It is clear that we thus assign to any $X \in \Gamma(E)$ a unique $i_X = \sum_j i_{X_j} \in {}^{-1}\mathrm{Der}\, \mathscr{A}$. As usual:

Lemma 1. *The sections in $\Gamma(E)$ are exactly the derivations in ${}^{-1}\mathrm{Der}\, \mathscr{A}$.*

Proof. Let $\delta \in {}^{-1}\mathrm{Der}\, \mathscr{A}$. Locally, in coordinates $v = (x, u) = (x^i, u^\alpha)$ over $U \subset M$, where the x^i are the base coordinates and the u^α the shifted fiber coordinates, this derivation reads

$$\delta|_U = \sum_\alpha f^\alpha(x) \partial_{u^\alpha} \in {}^{-1}\mathrm{Der}\, \mathscr{A}(U) .$$

The coordinates u^α of the fibers of the E_{-i} can be interpreted as frames of the E^*_{-i} over U. Let now u_α be the dual frames of the E_{-i} over U, $u_\alpha(u^\beta) = \delta^\beta_\alpha$, and consider

$$X_U := \sum_\alpha f^\alpha(x) u_\alpha \in \Gamma(U, E) \quad \text{and} \quad i_{X_U} \in {}^{-1}\mathrm{Der}\, \mathscr{A}(U) .$$

Since the actions of the derivations ∂_{u^α} and i_{u_α} on the generators of $\mathscr{A}(U)$ coincide, we have $\delta|_U = i_{X_U}$. It follows that the local sections $X_U \in \Gamma(U, E)$ defined over different chart domains U coincide in $U \cap V$ and thus define a global section $X \in \Gamma(E)$, such that $X|_U = X_U$. Finally, we obtain $\delta = i_X$. □

It is now quite easy to see which algebroid structure is encoded in the data of a split NQ-manifold. Let $E[\cdot] = \oplus_i E_{-i}[i]$ be a split N-manifold of degree $n \geq 1$ and let $Q \in \mathrm{Der}^1 \mathscr{A}$ be a homological vector field, $Q^2 = \frac{1}{2}[Q, Q] = 0$. For instance, in the case $n = 2$ and in local coordinates $(x^i, \xi^a, \eta^\mathfrak{a})$, such a derivation is of the type

$$Q = \sum f^i_a(x) \xi^a \partial_{x^i} + \sum (g^a_{bc}(x) \xi^b \xi^c + h^a_\mathfrak{a}(x) \eta^\mathfrak{a}) \partial_{\xi^a} + \sum (k^\mathfrak{a}_{abc}(x) \xi^a \xi^b \xi^c + \ell^\mathfrak{a}_{a\mathfrak{b}}(x) \xi^a \eta^\mathfrak{b}) \partial_{\eta^\mathfrak{a}} ; \quad (11)$$

it contains terms of homological degrees $0,1,2$. The considered NQ-manifold induces all the data required by the setup of the higher derived brackets method. Indeed, let

$$L = \mathrm{Der}^\bullet \mathscr{A} = \oplus_{\ell \geq -n} \mathrm{Der}^\ell \mathscr{A} = \oplus_{\ell \geq -n} \oplus_{s \geq -1} {}^s\mathrm{Der}^\ell \mathscr{A}$$

be the graded Lie algebra of derivations of $\mathscr{A}^\bullet = \oplus_k \mathscr{A}^k$ with bracket denoted by $[-,-]$. Observe that here we consider the standard grading, but that, see Proposition 1, that the space L is also graded by the homological degree. The graded commutator $[-,-]$ respects this homological degree as well. In fact, for $\delta \in {}^r\mathrm{Der}\,\mathscr{A}$ and $\delta' \in {}^s\mathrm{Der}\,\mathscr{A}$, we have

$$[\tilde{\varepsilon}, [\delta, \delta']] = [[\tilde{\varepsilon}, \delta], \delta'] + [\delta, [\tilde{\varepsilon}, \delta']] = (r+s)[\delta, \delta'] \,,$$

where $\tilde{\varepsilon} \in \mathrm{Der}^0 \mathscr{A}$ is the Euler field that encodes the homological degree of 'functions' and 'vector fields', see Section 3.1. Denote now by $P : L \to L$ the projector onto ${}^{-1}\mathrm{Der}\,\mathscr{A}$. Hence, $V := P(L) = {}^{-1}\mathrm{Der}\,\mathscr{A} \simeq \Gamma(E)$, see Lemma 1, is an abelian Lie subalgebra of L. Moreover, if $\delta = \sum_{r \geq -1} {}^r\delta \in L$ and $\delta' = \sum_{s \geq -1} {}^s\delta' \in L$, we have

$$P[\delta, \delta'] = [{}^{-1}\delta, {}^0\delta'] + [{}^0\delta, {}^{-1}\delta'] = P[P\delta, \delta'] + P[\delta, P\delta'] \,.$$

Let now $D := [Q, -]$ be the interior degree 1 derivation of L induced by Q. It respects the kernel $K := \mathrm{Ker}\,P = \oplus_{r \geq 0} {}^r\mathrm{Der}\,\mathscr{A}$. Indeed, $Q \in \mathrm{Der}^1 \mathscr{A} = \oplus_{s \geq -1} {}^s\mathrm{Der}^1 \mathscr{A}$ reads $Q = \sum_{s=0}^n {}^sQ$, see e.g. Equation (11), and, if $\kappa = \sum_{r \geq 0} {}^r\kappa \in K$, we get $D\kappa = \sum_r \sum_s [{}^sQ, {}^r\kappa] \in K$. As $D^2 = [Q, [Q, -]] = 0$, the higher derived brackets

$$\ell'_k(X_1, \ldots, X_k) = P[\ldots[[Q, X_1], X_2], \ldots, X_k]$$

provide a L_∞-antialgebra structure on $V = \Gamma(E)$. These k-ary brackets are actually given by

$$\ell'_k(X_1, \ldots, X_k) = P \sum_{s=0}^n [\ldots[[{}^sQ, X_1], X_2], \ldots, X_k] = [\ldots[[{}^{k-1}Q, X_1], X_2], \ldots, X_k] \,, \tag{12}$$

for $1 \leq k \leq n+1$, and they vanish otherwise. In view of Proposition 2, the brackets $\ell_k := s\,\ell'_k\,(s^{-1})^k$ endow $\Gamma(sE)$ with a Lie infinity structure.

To discover the algebroid structure encrypted in the homological vector field of an N-manifold, it remains to extract the information contained in the action of Q on the generators of degree 0, i.e. on $\mathscr{A}^0 = {}^0\mathscr{A} = C^\infty(M)$. Since, ${}^sQ : C^\infty(M) \to \Gamma(E^*_{-1}) \cap {}^s\mathscr{A}$, the derivation ${}^sQ$ vanishes on functions, if $s \neq 1$, whereas

$${}^1Q : C^\infty(M) \to \Gamma(E^*_{-1}) \,.$$

For any $X \in \Gamma(E_{-1})$ and any $f \in C^\infty(M)$, we set

$$\rho'(X)f := [{}^1Q, X]f = i_X\, {}^1Qf = -({}^1Qf)(X) \in C^\infty(M) \,. \tag{13}$$

As $[{}^1Q, X] \in {}^0\mathrm{Der}\,\mathscr{A}$ restricts to a derivation $[{}^1Q, X] \in \mathrm{Der}\,\mathscr{A}^0$, ρ' is a $C^\infty(M)$-linear map $\rho' : \Gamma(E_{-1}) \to \Gamma(TM)$ and can thus be viewed as a bundle map $\rho' : E_{-1} \to TM$. Moreover, if $X_j \in \Gamma(E)$ and $f \in C^\infty(M)$, the bracket

$$\ell'_k(X_1, \ldots, fX_j, \ldots, X_k) = [\ldots[\ldots[{}^{k-1}Q, X_1], \ldots, fX_j], \ldots, X_k]$$

can be computed as follows. It is easily seen that

$$[\ldots[{}^{k-1}Q, X_1], \ldots, fX_j] = \Big([\ldots[{}^{k-1}Q, X_1], \ldots, X_{j-1}]f\Big) \odot i_{X_j} + f[\ldots[{}^{k-1}Q, X_1], \ldots, X_j]$$

and

$$[\dots[{}^{k-1}Q,X_1],\dots,X_{j-1}]f = \pm i_{X_{j-1}}\dots i_{X_1}{}^{k-1}Qf\;.$$

If $k=2$ and $j=2$ (for $j=1$ it suffices to use the symmetry of the bracket), we thus find

$$\ell_2'(X_1,fX_2) = (i_{X_1}{}^1Qf)\,X_2 + f\ell_2'(X_1,X_2)\;.$$

The first term is nonzero only if $X_1\in\Gamma(E_{-1})$, in which case it reads $(\rho'(X_1)f)X_2$. If $k\neq 2$, we get

$$\ell_k'(X_1,\dots,fX_j,\dots,X_k) = f\ell_k'(X_1,\dots,X_j,\dots,X_k)\,,$$

as $X_m\simeq i_{X_m}$ is $C^\infty(M)$-linear. If we set now $\rho=\rho's^{-1}$, we obtain a bundle map $\rho:(sE)_0\to TM$, such that the brackets $\ell_i = s\ell_i'(s^{-1})^i$, $i\neq 2$, are $C^\infty(M)$-multilinear, whereas ℓ_2 is $C^\infty(M)$-bilinear if both arguments are elements of $\Gamma((sE)_{-i})$, $i\neq 0$, and verifies

$$\ell_2(x_0,fx) = f\ell_2(x_0,x) + (\rho(x_0)f)x\,,$$

for any $x_0\in\Gamma((sE)_0)$, $x\in\Gamma(sE)$, and $f\in C^\infty(M)$.

Remark 4. We thus recover the concept of split Lie n-algebroid introduced by Sheng and Zhu in [SZ11]. In fact, we proved that to any split NQ-manifold of degree n is associated a split Lie n-algebroid.

Definition 4. *A* split Lie n-algebroid*, $n\geq 1$, is a graded vector bundle $L=\oplus_{i=0}^{n-1}L_{-i}$ over a smooth manifold M, together with a bundle map $\rho:L_0\to TM$ and graded antisymmetric i-linear brackets $\ell_i:\Gamma(L)^{\times i}\to\Gamma(L)$, $i\in\{1,\dots,n+1\}$, of degree $2-i$, such that*

- *for any $r\geq 1$,*

$$\sum_{i+j=r+1}\;\sum_{\sigma\in\mathrm{Sh}(i,j-1)}(-1)^{i(j-1)}\chi(\sigma)\ell_j(\ell_i(x_{\sigma_1},\dots,x_{\sigma_i}),x_{\sigma_{i+1}},\dots,x_{\sigma_r}) = 0\,, \tag{14}$$

  where $x_1,\dots,x_r\in\Gamma(L)$, where $\mathrm{Sh}(i,j-1)$ denotes the set of $(i,j-1)$-shuffles, and where $\chi(\sigma)=\mathrm{sign}(\sigma)\varepsilon(\sigma)$ is the signature multiplied by the Koszul sign with respect to the grading of $\Gamma(L)$,

- *for $i\neq 2$, ℓ_i is $C^\infty(M)$-multilinear, whereas ℓ_2 is $C^\infty(M)$-bilinear if both arguments belong to $\Gamma(L_{-i})$, $i\neq 0$, and verifies*

$$\ell_2(x_0,fx) = f\ell_2(x_0,x) + (\rho(x_0)f)\,x\,,$$

  for any $x_0\in\Gamma(L_0)$, $x\in\Gamma(L)$, and $f\in C^\infty(M)$.

Observe that, since the graded vector bundle underlying a split Lie n-algebroid is concentrated in degrees $0,\dots,-n+1$, the Lie infinity algebra conditions (14) are nontrivial only for $1\leq r\leq n+2$. Indeed, if $r\geq n+3$, the degree of the LHS-terms is at most $2-i+2-j=3-r\leq -n$, so that all the terms vanish.

Examples 5. A Lie 1-algebroid is a Lie algebroid in the usual sense. Indeed, ℓ_1 vanishes, as it is of degree 1, and the Lie infinity algebra conditions reduce to the Jacobi identity. Further, a Lie n-algebroid over a point is exactly a Lie n-algebra, i.e. an n-term Lie infinity algebra.

Remark 5. As in the case of Lie infinity algebras, there exists a notion of *Lie n-antialgebroid*. Such an antialgebroid is made up by a graded vector bundle $K=\oplus_{i=1}^{n}K_{-i}$ over a manifold M, a bundle map $\rho':K_{-1}\to TM$, and a L_∞-antialgebra structure ℓ_i', $1\leq 1\leq n+1$, on $\Gamma(K)$, such that the ℓ_i' are

$C^\infty(M)$-multilinear for $i \neq 2$, whereas ℓ_2' is $C^\infty(M)$-bilinear if both arguments belong to $\Gamma(K_{-i})$, $i \neq 1$, and verifies

$$\ell_2'(x_1, fx) = f\ell_2'(x_1,x) + (\rho'(x_1)f)x \,,$$

for all $x_1 \in \Gamma(K_{-1})$, $x \in \Gamma(K)$, and $f \in C^\infty(M)$. To any Lie n-antialgebroid structure ℓ_i', ρ' on K corresponds a Lie n-algebroid structure $\ell_i = s\,\ell_i'\,(s^{-1})^i$, $\rho = \rho' s^{-1}$ on sK, and vice versa.

Remark 6. The orders of the brackets ℓ_i, viewed as differential operators, are comprehensible from the above construction of a Lie n-algebroid by means of derived brackets. Only the binary bracket ℓ_2 has an anchor, since ${}^sQ$ that implements ℓ_{s+1} vanishes on functions for $s \neq 1$. The reader might prefer an explanation in local coordinates. Consider first the case $n = 1$ of Lie algebroids. When dualizing the Lie bracket and the anchor in a local trivialization over a chart domain, we get the homological vector field

$$Q = \rho^i_a(x)\xi^a\partial_{x^i} + C^a_{bc}(x)\xi^b\xi^c\partial_{\xi^a} \,,$$

where x^i (resp., ξ^a) are the base coordinates (resp., the fiber coordinates), and where $\rho^i_a(x)$ (resp., $C^a_{bc}(x)$) are the components (resp., the structure functions) of the anchor (resp., of the bracket). Note that the anchor is thus encrypted in the terms in the ∂_{x^i}. If we pass to $n = 2$ and choose local coordinates $(x^i, \xi^a, \eta^{\mathfrak{a}})$, a homological vector field is, as mentioned above, of the type

$$Q = \sum f^i_a(x)\xi^a\partial_{x^i} + \sum(g^a_{bc}(x)\xi^b\xi^c + h^a_{\mathfrak{a}}(x)\eta^{\mathfrak{a}})\partial_{\xi^a} + \sum(k^{\mathfrak{a}}_{abc}(x)\xi^a\xi^b\xi^c + \ell^{\mathfrak{a}}_{a\mathfrak{b}}(x)\xi^a\eta^{\mathfrak{b}})\partial_{\eta^{\mathfrak{a}}} \,,$$

so that only ${}^1Q$ that defines ℓ_2 contains such terms.

3.4 Categories of Lie n-algebroids and NQ-manifolds: comparison of objects

Theorem 2. *There is a 1:1 correspondence between split NQ-manifolds (E,Q) of degree n and split Lie n-algebroids $(sE,(\ell_i)_i,\rho)$.*

Proof. Let $\ell_1,\ldots,\ell_{n+1},\rho$ be a Lie n-algebroid structure on a graded vector bundle $sE = (sE)_0 \oplus \ldots \oplus (sE)_{-n+1}$ over a manifold M. We will define on the degree n split N-manifold $E[\cdot] = \oplus_{i=1}^n E_{-i}[i]$ a derivation $Q \in \mathrm{Der}^1 \mathscr{A}$ (or, better, a global section of the sheaf of degree 1 derivations) that squares to 0. Remember that $\mathscr{A}$ (here, the algebra of global sections of the function sheaf) is given by

$$\mathscr{A} = \Gamma(\odot E^*)$$

$$= C^\infty(M) \oplus \Gamma(E^*_{-1}) \oplus \big(\Gamma(\odot^2 E^*_{-1}) \oplus \Gamma(E^*_{-2})\big) \oplus \big(\Gamma(\odot^3 E^*_{-1}) \oplus \Gamma(E^*_{-1} \odot E^*_{-2}) \oplus \Gamma(E^*_{-3})\big) \oplus \ldots$$

We first define the derivation Q on the generators $\omega_k \in \Gamma(E^*_{-k})$, $k \in \{1,\ldots,n\}$, and $\omega_0 \in C^\infty(M)$. We decompose $Q\omega_k \in \mathscr{A}^{k+1}$ with respect to the homological grading $\mathscr{A}^{k+1} = \oplus_{r=1}^{k+1}\, {}^r\!\mathscr{A}^{k+1}$ given by the number of generators. The component in ${}^r\!\mathscr{A}^{k+1}$ will be denoted by $Q^{k+1,r}\omega_k$. For instance,

$$Q^{4,2}\omega_3 \in \Gamma(E^*_{-1} \odot E^*_{-3}) \oplus \Gamma(\odot^2 E^*_{-2}) \,.$$

Hence, we have to define, for $X_1 \in \Gamma(E_{-1})$ and $X_2 \in \Gamma(E_{-3})$ or $X_1, X_2 \in \Gamma(E_{-2})$,

$$(Q^{4,2}\omega_3)(X_1,X_2) \in C^\infty(M) \,,$$

in a way that this function depend $C^\infty(M)$-bilinearly and graded symmetrically on its arguments. More generally, we define $(Q^{k+1,r}\omega_k)(X_1,\ldots,X_r) \in C^\infty(M)$, for all $X_j \in \Gamma(E_{-a_j})$ such that $\sum_j a_j = k+1$. We set

$$\rho' = \rho s, \quad \ell_r' = (-1)^{r(r-1)/2} s^{-1}\ell_r s^r \,, \tag{15}$$

so that ℓ'_r, ρ' provide a Lie n-antialgebroid structure on E. We now define

$$(Q^{1,1}\omega_0)(X_1) = -\rho'(X_1)\omega_0 \in C^\infty(M) \tag{16}$$

and, for $k \in \{1,\ldots,n\}$,

$$(Q^{k+1,r}\omega_k)(X_1,\ldots,X_r) = (-1)^k \omega_k(\ell'_r(X_1,\ldots,X_r)) \in C^\infty(M)\ , \tag{17}$$

if $r \in \{1,3,\ldots,k+1\}$, and

$$(Q^{k+1,2}\omega_k)(X_1,X_2) = (-1)^k \omega_k(\ell'_2(X_1,X_2)) - (\rho' \odot \omega_k)(X_1,X_2) \in C^\infty(M)\ , \tag{18}$$

if $r = 2$. The tensor product in the last equation is given by

$$(\rho' \odot \omega_k)(X_1,X_2) = (-1)^k \rho'(X_1)\omega_k(X_2) + (-1)^{a_1a_2+k}\rho'(X_2)\omega_k(X_1) \in C^\infty(M)\ .$$

Here (and in the following) we implicitly extend the anchor $\rho' : \Gamma(E_{-1}) \to \Gamma(TM)$ and $\omega_k : \Gamma(E_{-k}) \to C^\infty(M)$ by 0 to the whole module $\Gamma(E)$.

Graded symmetry is obvious and $C^\infty(M)$-multilinearity is nontrivial only for $r = 2$. Since

$$\ell'_2(X, fY) = f\ell'_2(X,Y) + (\rho'(X)f)Y \quad \text{and} \quad \ell'_2(fX,Y) = f\ell'_2(X,Y) + (-1)^X(\rho'(Y)f)X\ ,$$

for all $X,Y \in \Gamma(E)$, the function $(Q^{k+1,2}\omega_k)(X_1,X_2)$ is actually $C^\infty(M)$-bilinear.

To define Q on an arbitrary element $\omega = \sum_{k,s}\omega_{k,s} \in \Gamma(\odot E^*) = \mathscr{A} = \oplus_{k,s}\,{}^s\!\mathscr{A}^k$, we define the projections $Q^{k+1,r}\omega_{k,s}$ of $Q\omega_{k,s} \in \mathscr{A}^{k+1} = \oplus_{r=1}^{k+1}\,{}^r\!\mathscr{A}^{k+1}$ onto ${}^r\!\mathscr{A}^{k+1}$.

Definition 5. *Let ℓ_i, ρ be a Lie n-algebroid structure on sE and let $\ell'_i = (-1)^{i(i-1)/2}s^{-1}\ell_i s^i, \rho' = \rho\, s$ be the associated Lie n-antialgebroid data on E. The derivation $Q \in \mathrm{Der}^1\,\Gamma(\odot E^*)$, which is defined by*

$$Q^{k+1,r}\omega_{k,s} = (-1)^k \omega_{k,s} \circ (\ell'_{r-s+1} \odot \mathrm{id}_{s-1}) - \rho' \odot \omega_{k,s}\ , \tag{19}$$

where $\mathrm{id}_{s-1}(X_1,\ldots,X_{s-1}) = X_1 \odot \ldots \odot X_{s-1}$, *is the* Chevalley-Eilenberg differential *of the Lie n-algebroid sE.*

Equation (19) can be written more explicitly. For $k = 0$, we get

$$Q^{k+1,r}\omega_{k,s} = -\rho' \odot \omega_{k,s}\ , \tag{20}$$

for $k \geq 1$, if $r \geq s$, $r \neq s+1$,

$$Q^{k+1,r}\omega_{k,s} = (-1)^k\, \omega_{k,s} \circ (\ell'_{r-s+1} \odot \mathrm{id}_{s-1})\ , \tag{21}$$

if $r = s+1$,

$$Q^{k+1,r}\omega_{k,s} = (-1)^k\, \omega_{k,s} \circ (\ell'_2 \odot \mathrm{id}_{s-1}) - \rho' \odot \omega_{k,s}\ , \tag{22}$$

and if $r < s$,

$$Q^{k+1,r}\omega_{k,s} = 0\ . \tag{23}$$

Indeed, if $k = 0$, then $s = 0$ and $\mathrm{id}_{s-1} = 0$, so that (19) reduces to (20). Equations (21) and (22) are clear as well, as the term $\rho' \odot \omega_{k,s}$ can be evaluated only on $s+1$ sections of E (and must be interpreted as 0 on $r \neq s+1$ sections). For (23), it suffices to note that $\ell'_i = 0$, if $i \leq 0$.

Let us briefly comment on the definitions (20)-(23). Equation (20) is just a reformulation of (16). As for (21) and (22), note that the argument $\omega_{k,s}$ is an element of ${}^s\!\mathscr{A}^k$ and more precisely, say, of $\Gamma(E^*_{-c_1} \odot \ldots \odot E^*_{-c_s})$, $\sum_j c_j = k$, (for example $\Gamma(E^*_{-1} \odot E^*_{-3})$). Its image $Q^{k+1,r}\omega_{k,s} \in {}^r\!\mathscr{A}^{k+1}$ must be evaluated on $X_j \in \Gamma(E_{-a_j})$, $j \in \{1,\dots,r\}$, $\sum_j a_j = k+1$. The computation of $\ell'_{r-s+1} \odot \mathrm{id}_{s-1}$ on these X_j leads to terms each of which belongs to some $\Gamma(E_{-b_1}) \odot \dots \odot \Gamma(E_{-b_s})$, $\sum_j b_j = k$, since ℓ'_{r-s+1} has degree 1 (in the example, $\Gamma(E_{-1}) \odot \Gamma(E_{-3})$ or $\Gamma(E_{-2}) \odot \Gamma(E_{-2})$). In (21),(22) it is understood that the evaluation of $\omega_{k,s}$ on those terms that do not match is 0 by definition. Graded symmetry and $C^\infty(M)$-multilinearity are again straightforwardly checked.

To make Definition 5 meaningful (and to complete the proof), we still have to show that Q is a derivation and that $Q^2 = 0$.

As concerns the derivation property, observe that it follows from (8) and (19) that, for $\rho' = 0$, the endomorphism Q is actually a derivation. However, the map $\omega_{k,s} \mapsto \rho' \odot \omega_{k,s}$ is a derivation as well. Indeed, let $\eta_{\ell,t} \in {}^t\mathscr{A}^\ell$ and note that the tensor products in $\rho' \odot (\omega_{k,s} \odot \eta_{\ell,t})$ are defined differently. The first one is defined by means of vector fields that act on functions (we will use the notation L), the second by means of products of functions (notation $\cdot$). When omitting the arguments $X \in \Gamma(E)$ and the subscripts, we can write

$$\rho' \odot (\omega \odot \eta) = \sum \pm L_{\rho'}(\omega \cdot \eta) = \sum \pm (L_{\rho'}\omega) \cdot \eta + \sum \pm \omega \cdot (L_{\rho'}\eta)$$
$$= (\rho' \odot \omega) \odot \eta + (-1)^k \omega \odot (\rho' \odot \eta)\,.$$

Eventually, $Q \in \mathrm{Der}^1 \mathscr{A}$.

Below, we will explain that the Chevalley-Eilenberg complex of a Lie n-algebroid, Definition 5, 'reduces' for $n = 1$ to the Lie algebroid de Rham complex.

We prove now that $Q^2 = 0$, which holds true if it holds on the generators $\omega_k \in \Gamma(E^*_{-k})$, $1 \le k \le n$, and $\omega_0 \in C^\infty(M)$. To increase readability we work first up to sign. In the addendum to the proof, the interested reader can find the details about signs. It suffices to show that the sum

$$S = \sum_{s=1}^{k+1} (Q^{k+2,r} Q^{k+1,s} \omega_k)(X_1,\dots,X_r) \tag{24}$$

vanishes, for all $X_j \in \Gamma(E_{-a_j})$, such that $\sum_j a_j = k+2$, and each $1 \le r \le k+2$. Ignoring the signs, we get

$$S = \sum_{s=1}^{\inf(k+1,r)} (Q^{k+1,s}\omega_k)((\ell'_{r-s+1} \odot \mathrm{id}_{s-1})(X_1,\dots,X_r)) + (\rho' \odot (Q^{k+1,r-1}\omega_k))(X_1,\dots,X_r)\,.$$

In the preceding sum, we can replace $\inf(k+1,r)$ by r, since $Q^{k+1,k+2}\omega_k = 0$. Setting $t = r-s+1$, we then obtain

$$\begin{aligned}
S = &\sum_{s+t=r+1} \sum_{\sigma \in \mathrm{Sh}(t,s-1)} (Q^{k+1,s}\omega_k)(\ell'_t(X_{\sigma_1},\dots,X_{\sigma_t}), X_{\sigma_{t+1}},\dots,X_{\sigma_r}) \\
&+ \sum_i \rho'(X_i)(Q^{k+1,r-1}\omega_k)(X_1,\dots \hat{\imath} \dots,X_r) \\
= &\,\omega_k\left(\sum_{s+t=r+1} \sum_{\sigma \in \mathrm{Sh}(t,s-1)} \ell'_s(\ell'_t(X_{\sigma_1},\dots,X_{\sigma_t}), X_{\sigma_{t+1}},\dots,X_{\sigma_r}) \right) \\
&+ \sum_{\sigma \in \mathrm{Sh}(r-1,1)} (\rho' \odot \omega_k)(\ell'_{r-1}(X_{\sigma_1},\dots,X_{\sigma_{r-1}}), X_{\sigma_r}) \\
&+ \sum_i \rho'(X_i) \left(\omega_k(\ell'_{r-1}(X_1,\dots \hat{\imath} \dots,X_r)) + \delta_{r,3}(\rho' \odot \omega_k)(X_1,\dots \hat{\imath} \dots,X_r) \right)\,,
\end{aligned} \tag{25}$$

where the first term vanishes in view of the L_∞-antialgebra condition (10). Hence,

$$S = \sum_i \rho'(\ell'_{r-1}(X_1,\ldots\hat{i}\ldots,X_r))\omega_k(X_i) + \sum_i \rho'(X_i)\omega_k(\ell'_{r-1}(X_1,\ldots\hat{i}\ldots,X_r))$$
$$+ \sum_i \rho'(X_i)\omega_k(\ell'_{r-1}(X_1,\ldots\hat{i}\ldots,X_r)) + \delta_{r,3}\sum_i \rho'(X_i)\left((\rho'\odot\omega_k)(X_1,\ldots\hat{i}\ldots,X_r)\right) . \tag{26}$$

When taking signs into account, we see that the second and third sums cancel out. If no a_i is equal to k, the first and fourth sums vanish as well. Otherwise, $a_1+\ldots\hat{i}\ldots+a_r=2$ and $r=2$ or $r=3$. If $r=2$, the first sum reads

$$\rho'(\ell'_1(X_2))\omega_k(X_1) + \rho'(\ell'_1(X_1))\omega_k(X_2) , \tag{27}$$

where $a_1=k$ and $a_2=2$ or vice versa. It thus suffices to show that

$$\rho'\circ\ell'_1 = 0 \tag{28}$$

on $\Gamma(E_{-2})$. This conclusion follows from the L_∞-condition

$$\ell'_1(\ell'_2(X,fY)) + \ell'_2(\ell'_1(X),fY) + (-1)^X\ell'_2(X,\ell'_1(fY)) = 0 ,$$

written for $X\in\Gamma(E_{-2})$. If $r=3$, the first and fourth sums exist, so that

$$S = \rho'(\ell'_2(X_2,X_3))\omega_k(X_1) + \ldots + \rho'(X_2)\left(\rho'(X_1)\omega_k(X_3) + \rho'(X_3)\omega_k(X_1)\right) + \ldots , \tag{29}$$

where one of the a_i is equal to k and the two others equal to 1. It is easily seen that it suffices to prove that

$$\rho'(\ell'_2(X,Y)) = \rho'(X)\rho'(Y) - \rho'(Y)\rho'(X) , \tag{30}$$

for all $X,Y\in\Gamma(E_{-1})$. The result is encoded in the L_∞-condition for brackets ℓ'_i,ℓ'_j, $i+j=4$, written for X,Y,fZ, with $X,Y\in\Gamma(E_{-1})$. This is straightforwardly checked (we actually obtain the same property for ρ and ℓ_2). This completes the construction of an NQ-manifold from a Lie n-algebroid.

Conversely, we can construct a Lie n-algebroid from an NQ-manifold (E,Q). Indeed, the definitions (16)-(18) can easily be inverted. Equation (16) defines the anchor ρ' from Q. Let now $X_j\in\Gamma(E_{-a_j})$, $1\le j\le r$, set $k:=\sum a_j - 1$, and let $\omega_k\in\Gamma(E^*_{-k})$. Equation (17) gives, for $r\neq 2$,

$$\left(\ell'_r(X_1,\ldots,X_r)\right)(\omega_k) = (Q^{k+1,r}\omega_k)(X_1,\ldots,X_r) ,$$

since $(-1)^{k(k+1)}=1$. Equation (18) provides $\left(\ell'_2(X_1,X_2)\right)(\omega_k)$. Clearly, ρ' coincides with the anchor (13), say ρ'', defined in the construction of a Lie n-algebroid via higher derived brackets. Moreover, if we denote the higher brackets (12) by ℓ''_r, we have

$$\ell'_r = (-1)^r\ell''_r . \tag{31}$$

Indeed, when computing

$$\left(\ell''_r(X_1,\ldots,X_r)\right)(\omega_k) = [\ldots[[{}^{r-1}Q,X_1],X_2],\ldots,X_r](\omega_k) ,$$

we get terms of the type $i_{X_{\sigma_1}}\ldots i_{X_{\sigma_j}}\,{}^{r-1}Q\,i_{X_{\sigma_{j+1}}}\ldots i_{X_{\sigma_r}}\omega_k$. However, if j differs from r and $r-1$, such a term vanishes. Even for $j=r-1$, it vanishes, except if $a_{\sigma_r}=k$, in which case we have $r=2$, since $\sum a_j = k+1$. If $r\neq 2$, the derived bracket ℓ''_r is given by a unique term. It suffices to compute the sign of this interior product and to insert the sections X_j into ${}^{r-1}Q\omega_k$, i.e., if we change notation, into $Q^{k+1,r}\omega_k$, which generates new signs. Combining all these signs, we actually get $(-1)^r$. If $r=2$, the bracket ℓ''_2 contains three terms. The proof is just a matter of computation. Since ℓ''_r,ρ'' define a Lie n-antialgebroid structure on E, the same holds obviously true for ℓ'_r,ρ', so that, to complete the proof, it suffices to consider the associated Lie n-algebroid $(sE,(\ell_r)_r,\rho)$.

The constructions of a higher Lie algebroid from a higher Q-manifold and vice versa are of course inverses of each other. □

Addendum. The sign in the first term of (25) is $-\varepsilon(\sigma)$. If we denote, for simplicity, the degree of X_j by X_j instead of $-a_j$, those in the four terms of (26) are

$$(-1)^{X_i(X_{i+1}+\ldots+X_r)+k(X_1+\ldots\hat{i}\ldots+X_r)}, (-1)^{k+X_i(X_1+\ldots+X_{i-1}+k+1)},$$
$$(-1)^{k+1+X_i(X_1+\ldots+X_{i-1}+k+1)}, (-1)^{X_i(X_1+\ldots+X_{i-1}+k+1)}.$$

It is thus clear that the first term of (25) vanishes and that the second and third terms of (26) cancel. Moreover, (27) vanishes in view of (28), independently of the involved signs. When writing explicitly the terms of (29), we get for instance

$$(-1)^{(X_1+k)(X_2+X_3)}\left[\rho'(\ell_2'(X_2,X_3))\omega_k(X_1)\right.$$
$$\left.+(-1)^{X_2}\rho'(X_2)\rho'(X_3)\omega_k(X_1)+(-1)^{(X_2+1)X_3}\rho'(X_3)\rho'(X_2)\omega_k(X_1)\right].$$

It now suffices to observe that this sum vanishes, if $X_1\neq -1$ or $X_2\neq -1$, and that otherwise it reads

$$\rho'(\ell_2'(X_2,X_3))\omega_k(X_1)-\rho'(X_2)\rho'(X_3)\omega_k(X_1)+\rho'(X_3)\rho'(X_2)\omega_k(X_1)$$

and thus vanishes in view of (30).

Remark 7. The preceding proof shows two facts:

- For any split Lie n-algebroid $(L,(\ell_r)_r,\rho)$ over a manifold M, the bundle map $\rho : L_0\to TM$ verifies
$$\rho(\ell_2(X,Y)) = [\rho(X),\rho(Y)]\,,$$
for all $X,Y\in\Gamma(L_0)$, where $[-,-]$ is the bracket of vector fields. In other words, the n-algebroid anchor is a representation on $\mathrm{Vect}(M)$ of the Lie algebra (up to homotopy) bracket ℓ_2 on $\Gamma(L_0)$.
- Any Lie n-algebroid $(L,(\ell_r)_r,\rho)$ is implemented by higher derived brackets. Indeed, let Q be the homological vector field associated to the Lie n-antialgebroid structure $\ell_r' = (-1)^{r(r-1)/2}s^{-1}\ell_r s^r$, $\rho'=\rho s$. From Q we construct via higher derived brackets the antialgebroid structure ℓ_r'',ρ'', and we reconstruct ℓ_r',ρ'. Hence, $\ell_r = s\ell_r'(s^{-1})^r = (-1)^r s\ell_r''(s^{-1})^r$ and $\rho=\rho's^{-1}=\rho''s^{-1}$.

Remark 8. For $n=1$, i.e. in the Lie algebroid case, the Chevalley-Eilenberg differential (19) coincides with the de Rham differential of the considered Lie algebroid. More precisely, the shifting operator allows to interpret the Chevalley-Eilenberg differential $Q\in \mathrm{Diff}^1\,\Gamma(\odot E^*)$ of the Lie n-algebroid $(sE,(\ell_r)_r,\rho)$ as differential $\tilde{Q}$ on $\Gamma(\boxdot(sE)^*)$. The computation is technical and will not be given here. If $\eta_{k,s}\in\Gamma((sE)^*_{-a_1+1}\boxdot\ldots\boxdot(sE)^*_{-a_s+1})$, $\sum a_j = k$, we find

$$\tilde{Q}^{k+1,r}\eta_{k,s} = (-1)^{(r-s+1)(s-1)}\eta_{k,s}\circ(\ell_{r-s+1}\boxdot \mathrm{id}_{s-1})-\rho\boxdot\eta_{k,s}\,, \tag{32}$$

where $\mathrm{id}_{s-1}(X_1,\ldots,X_{s-1}) = X_1\boxdot\ldots\boxdot X_{s-1}$. In the case $n=1$, necessarily $s=k, r=k+1$, and $\eta_{k,s}=:\eta_k\in\Gamma(\wedge^k(sE)^*_0)$. It is easily seen that Equation (32) then reduces to the usual de Rham cohomology operator.

4 Geometry of Lie n-algebroid morphisms

4.1 General morphisms of Lie n-algebroids

In this section, we define morphisms between Lie n-algebroids over different bases in terms of anchors and brackets. In the case $n = 1$, we recover the notion of Lie algebroid morphism [Mac05], and for n-algebroids over a point, the new concept coincides with that of Lie infinity algebra morphism.

Let $E = \oplus_{i=1}^n E_{-i}$ (resp., $F = \oplus_{i=1}^n F_{-i}$) be a graded vector bundle over M (resp., N). A graded vector bundle morphism (in the categorical sense, i.e. a vector bundle morphism of degree 0) $\phi'_r : \odot^r E \to F$, $r \geq 1$, is a smooth map over a smooth map $\phi_0 : M \to N$, with linear restrictions to the fibers. For instance,

$$\phi'_2 : \wedge^2 E_{-1,x} \to F_{-2,\phi_0(x)},\ \phi'_2 : E_{-1,x} \otimes E_{-2,x} \to F_{-3,\phi_0(x)}, \cdots$$

are linear. Remark that if $r \geq n+1$, the highest degree in $\odot^r E$ is $-r \leq -n-1 < -n$, so that ϕ'_r is necessarily zero. A graded vector bundle morphism $\phi'_r : \odot^r E \to F$ can be viewed as a vector bundle morphism $\phi_r : \boxdot^r sE \to sF$ of degree $1-r$:

$$\phi_r = s\,\phi'_r (s^{-1})^r \quad \text{and} \quad \phi'_r = (-1)^{r(r-1)/2} s^{-1} \phi_r s^r\ .$$

If $X_i \in \Gamma(E_{-a_i})$, $i \in \{1, \ldots, r\}$, then $\phi'_r \circ X := \phi'_r \circ (X_1 \odot \ldots \odot X_r)$ has obviously a decomposition of the form

$$\phi'_r \circ X = \sum_j f_j^X\ \xi_j^X \circ \phi_0\ , \tag{33}$$

where the sum is finite, $f_j^X \in C^\infty(M)$ and $\xi_j^X \in \Gamma(F_{-\sum a_i})$. Indeed, it suffices to take as ξ_j^X a finite generating family of sections in the $C^\infty(N)$-module $\Gamma(F_{-\sum a_i})$. Furthermore, it is easily seen that the graded symmetric tensor product $\phi'_{t_1} \odot \ldots \odot \phi'_{t_r}$ of graded vector bundle morphisms is given as follows. If $X_i \in \Gamma(E_{-a_i})$, $i \in \{1, \ldots, t\}$, and $t_1 + \ldots + t_r = t$, $t_j \neq 0$, then

$$(\phi'_{t_1} \odot \ldots \odot \phi'_{t_r}) \circ (X_1, \ldots, X_t) = \sum_{\sigma \in \mathrm{Sh}(t_1,\ldots,t_r)} \sum_{j_1} \ldots \sum_{j_r} \varepsilon(\sigma)\, f_{j_1}^{X^{\sigma^1}} \ldots f_{j_r}^{X^{\sigma^r}} (\xi_{j_1}^{X^{\sigma^1}} \odot \ldots \odot \xi_{j_r}^{X^{\sigma^r}}) \circ \phi_0\ , \tag{34}$$

where $\varepsilon(\sigma)$ is the Koszul sign.

Let ℓ_i, ρ (resp., m_i, r) be a Lie n-algebroid structure on sE (resp., sF), and denote by ℓ'_i, ρ' (resp., m'_i, r') the corresponding Lie n-antialgebroid structure on E (resp., F).

Definition 6. *A* morphism of Lie n-algebroids *between sE and sF is a family $\phi_r : \boxdot^r sE \to sF$, $1 \leq r \leq n$, of degree $1-r$ vector bundle morphisms over a base map $\phi_0 : M \to N$, such that*

$$r' \circ \phi'_1 = T\phi_0 \circ \rho'\ , \tag{35}$$

as well as, for any $1 \leq t \leq n+1$ and any homogeneous sections X_i of E, $i \in \{1, \ldots, t\}$, with decompositions

$$\phi'_r \circ X_I = \sum_j f_j^{X_I}\ \xi_j^{X_I} \circ \phi_0$$

(for any r and any product $X_I := X_{i_1} \odot \ldots \odot X_{i_r}$),

$$\sum_{r+s=t+1} \sum_{\sigma \in \mathrm{Sh}(s,r-1)} \varepsilon(\sigma)\ \phi'_r \circ (\ell'_s(X_{\sigma_1}, \ldots, X_{\sigma_s}) \odot X_{\sigma_{s+1}} \odot \ldots \odot X_{\sigma_t})$$

$$+ \sum_{ij} (-1)^{\tilde{X}_i(\tilde{X}_1 + \ldots + \tilde{X}_{i-1})+1}\ (\rho'(X_i) f_j^{X_1 \ldots \hat{\imath} \ldots X_t}) \xi_j^{X_1 \ldots \hat{\imath} \ldots X_t} \circ \phi_0$$

$$= \sum_{r=1}^{t} \frac{1}{r!} \sum_{\substack{t_1+\ldots+t_r=t \\ t_j \neq 0}} \sum_{\sigma \in \mathrm{Sh}(t_1,\ldots,t_r)} \sum_{j_1} \ldots \sum_{j_r} \varepsilon(\sigma) f_{j_1}^{X^{\sigma^1}} \ldots f_{j_r}^{X^{\sigma^r}} m'_r(\xi_{j_1}^{X^{\sigma^1}}, \ldots, \xi_{j_r}^{X^{\sigma^r}}) \circ \phi_0 \,, \tag{36}$$

where $\tilde{X}_k$ denotes the degree of X_k.

Note that for $t \geq n+2$, the highest degree of the terms in Equation (36) is $1-t \leq -n-1 < -n$, so that any term necessarily vanishes.

Remark 9.

- This definition is the geometric translation of the natural supergeometric / algebraic definition of Lie n-algebroid morphisms, see below.
- For $n=1$, the definition reduces to that of morphisms of Lie algebroids over different bases, see [HM90], [BKS04], [Mac05].

  Indeed, note first that for $n=1$, the maps ϕ'_r, $r \neq 1$, vanish, as they are of degree 0. We already noticed that the same is true for ℓ'_r, m'_r, $r \neq 2$.

  For $t \neq 2$, Condition (36) is trivial. To understand this claim, observe that the sum in the second row of (36) (resp., the RHS of (36)) is constructed from the decomposition (33) of $\phi'_{t-1} \circ (X_1 \odot \ldots \hat{\imath} \ldots \odot X_t)$ (resp., the decomposition (34) of $(\phi'_{t_1} \odot \ldots \odot \phi'_{t_r}) \circ (X_1, \ldots, X_t)$). It is now clear that the general term of the sum in the first row of (36) is nonzero only if $r=1$ and $s=2$, hence if $t=2$; that the sum in the second row does not vanish only if $t=2$; that the RHS does not vanish only if $r=2$ and $t_1=t_2=1$, hence, if $t=2$.

  Eventually, for $t=2$, Equation (36) is easily written in terms of $\phi_1, \ell_2, \rho, m_2$. It then coincides with the similar condition in the aforementioned works.
- A priori Definition 6 depends on the choice of the involved decompositions. However, it is known, at least in the Lie algebroid case $n=1$, that all the terms are well-defined, see [BKS04], [Mac05]. For $n>1$, this fact is a consequence of Theorem 3, see below.

Before continuing, we work out an equivalent version of the anchor condition (35), which uses the decomposition (33). Let $g \in C^\infty(N)$, let $X \in \Gamma(E_{-1})$, and let all the other objects be as above. Remember first that, if $Z_x \in T_xM$, $x \in M$, we have $Z_x(g \circ \phi_0) = (d_{\phi_0(x)}g)((T_x\phi_0)Z_x) = ((T_x\phi_0)Z_x)(g)$, and that, if $Y \in \mathrm{Vect}(N)$, we get $Y_{\phi_0(x)}g = (Yg)(\phi_0(x)) = (\phi_0^*(Yg))(x)$. Assume now that

$$\phi'_1 \circ X = \sum_j f_j^X \xi_j^X \circ \phi_0 \,. \tag{37}$$

When using the just recalled results and taking into account the decomposition (37), we see that Equation (35) is equivalent to the equation

$$\big(\rho'(X)(\phi_0^* g)\big)_x = \rho'(X_x)(g \circ \phi_0) = ((T_x\phi_0)(\rho'(X_x)))(g) = r'(\phi'_1 X_x)(g)$$

$$= \sum_j f_j^X(x) r'(\xi_j^X)_{\phi_0(x)}(g) = \left(\sum_j f_j^X \phi_0^*(r'(\xi_j^X) g)\right)(x) \,. \tag{38}$$

4.2 Base-preserving morphisms of Lie n-algebroids

If $\phi_0 : M \to N$ is a diffeomorphism, the Lie n-algebroid morphism conditions can be simplified. Indeed, identify the manifolds M and N, so that $\phi_0 = \mathrm{id}$.

The anchor condition (35) then reduces to

$$r' \circ \phi'_1 = \rho' \;, \tag{39}$$

which is equivalent to $r'(\phi'_1 \circ X) = \rho'(X)$, for all $X \in \Gamma(E)$, provided we define ρ' and r' by 0 in all degrees different from -1.

As for the condition (36), let us work – to simplify – up to sign. Remember first that m'_r, $r \neq 2$, is $C^\infty(N)$-multilinear and that m'_2 verifies, for any $f,g \in C^\infty(N)$ and any $X,Y \in \Gamma(F)$,

$$fg\, m'_2(X,Y) = m'_2(fX,gY) + f(r'(X)g)Y + g(r'(Y)f)X,$$

where we use again the just mentioned extension of r' by 0. The anchor terms in the LHS of (36) then read

$$\sum_{i,\ell} (r'(\phi'_1 \circ X_i) f_\ell^{X_{\hat{\imath}}}) \xi_\ell^{X_{\hat{\imath}}} \;, \tag{40}$$

where $X_{\hat{\imath}} = X_1 \ldots \hat{\imath} \ldots X_t$. In view of Equation (34), the RHS of (36) is given by

$$\sum_{r=1}^{t} \frac{1}{r!} \sum_{\substack{t_1+\ldots+t_r=t \\ t_j \neq 0}} m'_r \left((\phi'_{t_1} \odot \ldots \odot \phi'_{t_r}) \circ (X_1, \ldots, X_t) \right) + \ldots \;, \tag{41}$$

where $\ldots$ denote the anchor terms that appear if $r = 2$.

If $t = 1$, there are no such terms; on the other hand, the sum (40) then vanishes (we will refer to this observation as result $(\star)$). Assume in the following that $t \geq 2$. The potential anchor terms are generated by the transformation of the sum

$$\frac{1}{2} \sum_{\substack{t_1+t_2=t \\ t_i \neq 0}} \sum_{\sigma \in \mathrm{Sh}(t_1,t_2)} \sum_{j_1} \sum_{j_2} f_{j_1}^{X^{\sigma^1}} f_{j_2}^{X^{\sigma^2}} m'_2(\xi_{j_1}^{X^{\sigma^1}}, \xi_{j_2}^{X^{\sigma^2}}) \;.$$

If the total degree of $X_{\sigma_1}, \ldots, X_{\sigma_{t_1}}$ and the total degree of $X_{\sigma_{t_1+1}}, \ldots, X_{\sigma_{t_1+t_2}}$ differ both from -1, no anchor terms appear. Otherwise, $t_1 = 1$ (and $t_2 = t-1$) or $t_2 = 1$ (and $t_1 = t-1$). These possibilities correspond to different terms in the sum over t_1, t_2 if and only if $t \geq 3$.

Let now $t \geq 3$. In view of what has been said, additional terms appear only in the two mentioned cases. They are given by

$$\begin{aligned} \frac{1}{2} 2 \sum_{i,j,\ell} \left(f_j^{X_i} (r'(\xi_j^{X_i}) f_\ell^{X_{\hat{\imath}}}) \xi_\ell^{X_{\hat{\imath}}} + f_\ell^{X_{\hat{\imath}}} (r'(\xi_\ell^{X_{\hat{\imath}}}) f_j^{X_i}) \xi_j^{X_i} \right) &= \sum_{i,j,\ell} f_j^{X_i} (r'(\xi_j^{X_i}) f_\ell^{X_{\hat{\imath}}}) \xi_\ell^{X_{\hat{\imath}}} \\ &= \sum_{i,\ell} (r'(\phi'_1 \circ X_i) f_\ell^{X_{\hat{\imath}}}) \xi_\ell^{X_{\hat{\imath}}} \;, \end{aligned} \tag{42}$$

since the degree of $\xi_\ell^{X_{\hat{\imath}}}$ is < -1.

If $t = 2$, the sum over t_1, t_2 contains a unique term $t_1 = t_2 = 1$ and the anchor terms (although possibly zero) read

$$\frac{1}{2} 2 \sum_{j,\ell} \left(f_j^{X_1} (r'(\xi_j^{X_1}) f_\ell^{X_2}) \xi_\ell^{X_2} + f_j^{X_2} (r'(\xi_j^{X_2}) f_\ell^{X_1}) \xi_\ell^{X_1} \right)$$

$$= \sum_{\ell} \Big((r'(\phi_1' \circ X_1) f_\ell^{X_2}) \xi_\ell^{X_2} + (r'(\phi_1' \circ X_2) f_\ell^{X_1}) \xi_\ell^{X_1} \Big)$$

$$= \sum_{i,\ell} (r'(\phi_1' \circ X_i) f_\ell^{X_i}) \xi_\ell^{X_i} \; . \tag{43}$$

Since the sums (40) and (42) or (43) cancel out (see also ($\star$)), the simplified form of the algebroid morphism condition (36) follows. Hence, the next reformulation.

Definition 7. *Let sE and sF be two Lie n-algebroids over a same base. A* base-preserving morphism of Lie n-algebroids *between sE and sF is a family $\phi_r : \boxdot^r sE \to sF$, $1 \le r \le n$, of degree $1-r$ vector bundle morphisms (over the identity) that verify the condition*

$$r' \circ \phi_1' = \rho' \; , \tag{44}$$

as well as, for any $1 \le t \le n+1$ and any homogeneous sections X_i of E, $i \in \{1,\ldots,t\}$, the condition

$$\sum_{r+s=t+1} \sum_{\sigma \in \mathrm{Sh}(s,r-1)} \varepsilon(\sigma) \phi_r' \circ \big(\ell_s'(X_{\sigma_1},\ldots,X_{\sigma_s}) \odot X_{\sigma_{s+1}} \odot \ldots \odot X_{\sigma_t} \big)$$

$$= \sum_{r=1}^{t} \frac{1}{r!} \sum_{\substack{t_1+\ldots+t_r=t \\ t_j \neq 0}} m_r' \big((\phi_{t_1}' \odot \ldots \odot \phi_{t_r}') \circ (X_1,\ldots,X_t) \big) \; . \tag{45}$$

Remark 10. Remember that a Lie n-algebroid over a point is exactly a Lie n-algebra, hence a truncated Lie infinity algebra.

When rewriting the condition (45) in terms of ϕ_r, ℓ_r, and m_r, we obtain

$$\sum_{r+s=t+1} \sum_{\sigma \in \mathrm{Sh}(s,r-1)} (-1)^{s(r-1)} \operatorname{sign}\sigma\, \varepsilon(\sigma) \phi_r \circ \big(\ell_s(Y_{\sigma_1},\ldots,Y_{\sigma_s}), Y_{\sigma_{s+1}},\ldots,Y_{\sigma_t} \big)$$

$$\sum_{r=1}^{t} \frac{1}{r!} \sum_{\substack{t_1+\ldots+t_r=t \\ t_j \neq 0}} \sum_{\sigma \in \mathrm{Sh}(t_1,\ldots,t_r)} \pm \operatorname{sign}\sigma\, \varepsilon(\sigma)\, m_r(\phi_{t_1} \circ Y^{\sigma^1},\ldots,\phi_{t_r} \circ Y^{\sigma^r}) \; , \tag{46}$$

where we wrote Y_i instead of sX_i and where

$$\pm = (-1)^{r(r-1)/2+\sum_j t_j(r-j)} + \sum_j |Y^{\sigma^j}|(r-j+t_{j+1}+\ldots+t_r) \; ,$$

$|Y^{\sigma^j}|$ being the sum of the degrees of the components of Y^{σ^j}. This is exactly the Lie infinity algebra morphism condition, see [Sch04], [AP10], [LV11]. Hence, the definition of base-preserving Lie n-algebroid morphisms coincides over a point (bundles become spaces, bundle morphisms become linear maps, anchors vanish, sections become vectors and compositions evaluations) with the definition of (truncated) Lie infinity algebra morphisms. We thus prove a result conjectured in [SZ11], Remark 2.5.

4.3 Categories of Lie n-algebroids and NQ-manifolds: comparison of morphisms

In this section we show that morphisms of split Lie n-algebroids are morphisms of NQ-manifolds between split NQ-manifolds.

Proposition 3. *There is a 1-to-1 correspondence between families $\phi_r : \boxdot^r sE \to sF$, $1 \le r \le n$, of degree $1-r$ vector bundle morphisms over a map ϕ_0, and graded algebra morphisms $\Phi : \Gamma(\odot F^*) \to \Gamma(\odot E^*)$.*

Proof. To define $\Phi:\Gamma(\odot F^*)\to\Gamma(\odot E^*)$, we define, for $\eta_{k,s}\in\Gamma(F^*_{-b_1}\odot\ldots\odot F^*_{-b_s})$, $\sum b_j=k$, the projection of $\Phi^{k,r}\eta_{k,s}$ onto any $\Gamma(E^*_{-a_1}\odot\ldots\odot E^*_{-a_r})$, $\sum a_j=k$. More precisely, we define $(\Phi^{k,r}\eta_{k,s})(X_1,\ldots,X_r)\in C^\infty(M)$, $X_j\in\Gamma(E_{-a_j})$, in a way such that the dependence on the X_j be $C^\infty(M)$-multilinear and graded symmetric.

We first set

$$\Phi^{0,0}:C^\infty(N)\ni g\mapsto g\circ\phi_0\in C^\infty(M)\;. \tag{47}$$

Then, for $k\geq 1$, we define $(\Phi^{k,r}\eta_{k,s})(X_1,\ldots,X_r)$ by 0, if $s>r$, and set, for $s\leq r$,

$$(\Phi^{k,r}\eta_{k,s})(X_1,\ldots,X_r)=\langle\eta_{k,s}\circ\phi_0,\frac{1}{s!}\sum_{\substack{r_1+\ldots+r_s=r\\ r_i\neq 0}}(\phi'_{r_1}\odot\ldots\odot\phi'_{r_s})\circ(X_1,\ldots,X_r)\rangle\;. \tag{48}$$

Indeed, for any $x\in M$, we have

$$\frac{1}{s!}\sum_{\substack{r_1+\ldots+r_s=r\\ r_i\neq 0}}(\phi'_{r_1}\odot\ldots\odot\phi'_{r_s})(X_{1,x},\ldots,X_{r,x})$$

$$=\frac{1}{s!}\sum_{\substack{r_1+\ldots+r_s=r\\ r_i\neq 0}}\;\sum_{\sigma\in\mathrm{Sh}(r_1,\ldots,r_s)}\varepsilon(\sigma)\;\phi'_{r_1}(X_x^{\sigma^1})\odot\ldots\odot\phi'_{r_s}(X_x^{\sigma^s})\;, \tag{49}$$

where a notation as $X_x^{\sigma^1}$ means $X_{\sigma_1,x},\ldots,X_{\sigma_{r_1},x}$. If we denote the sum of the degrees $-a_{\sigma_j}$ of these $X_{\sigma_j,x}$ by $|X_x^{\sigma^1}|$, we get

$$\phi'_{r_1}(X_x^{\sigma^1})\odot\ldots\odot\phi'_{r_s}(X_x^{\sigma^s})\in F_{|X_x^{\sigma^1}|,\phi_0(x)}\odot\ldots\odot F_{|X_x^{\sigma^s}|,\phi_0(x)}\;.$$

On the other hand, $\eta_{k,s;\phi_0(x)}$ is an element of $\left(F_{-b_1;\phi_0(x)}\odot\ldots\odot F_{-b_s;\phi_0(x)}\right)^*$. Of course, the contraction of the terms of the RHS of (49) with $\eta_{k,s;\phi_0(x)}$ gives a nonzero contribution only if the considered term belongs to the source space of $\eta_{k,s;\phi_0(x)}$. It is now clear that the RHS of (48) is a function on M that depends on the X_j in a $C^\infty(M)$-multilinear and graded symmetric way.

The definition of $\Phi:\Gamma(\odot F^*)\to\Gamma(\odot E^*)$ is now complete. In view of (9) and (47), (48), Φ is a graded algebra (GA) morphism.

Remark 11. It is easily checked that, for $n=1$, $E=TM$, $F=TN$ and $\phi_1=T\phi_0$, the algebra morphism Φ is just the pullback $\phi_0^*:\Gamma(\wedge T^*N)\to\Gamma(\wedge T^*M)$ of differential forms by ϕ_0.

Proof (continuation). Conversely, to any GA morphism $\Phi:\Gamma(\odot F^*)\to\Gamma(\odot E^*)$ we can associate a family $\phi'_r:\odot^r E\to F$, $r\geq 1$, of graded vector bundle morphisms over a map ϕ_0.

The map Φ is in particular an associative algebra morphism $\Phi:C^\infty(N)\to C^\infty(M)$. Hence, it is the pullback by a smooth map $\phi_0:M\to N$, see e.g. [AMR83], [Bko65]. It follows that, for any $g\in C^\infty(N)$ and $\eta\in\Gamma(\odot F^*)$,

$$\Phi(g\eta)=(g\circ\phi_0)(\Phi\eta)\;.$$

This 'function-linearity' implies as usual that Φ is local, i.e. that $\Phi\eta=0$ on $\phi_0^{-1}(V)$, if $\eta=0$ on V, where V is an open subset of N (indeed, for any $x\in\phi_0^{-1}(V)$, consider a bump function α around $\phi_0(x)$, and note that $\Phi\eta=\Phi((1-\alpha)\,\eta)$). In fact, for any $x\in M$, we even have $(\Phi\eta)_x=0$, if $\eta_{\phi_0(x)}=0$ (indeed, take a local frame $(b_i)_i$ of $\odot F^*$ in $V\ni\phi_0(x)$ and set $\eta=\sum_i s^ib_i$ in V; if the bump function α is as above and has value 1 in $W\ni\phi_0(x)$, then $\eta=\sum_i(\alpha s^i)(\alpha b_i)$ in W; due to locality, $(\Phi\eta)_x=\sum_i(\alpha s^i)_{\phi_0(x)}(\Phi(\alpha b_i))_x=0$): the value $(\Phi\eta)_x$, $x\in M$, only depends on the value $\eta_{\phi_0(x)}$.

To define, for $r \geq 1$ and $x \in M$, a linear map $\phi'_r : \odot^r E_x \to F_{\phi_0(x)}$ of degree 0, associate to any $p \in E_{-a_1,x} \odot \ldots \odot E_{-a_r,x} \subset \odot^r E_x$, $\sum a_j = k$, a unique

$$\phi'_r(p) \in F_{-k,\phi_0(x)} \simeq (F^*_{-k,\phi_0(x)})^* \ .$$

Hence, let $q^* \in F^*_{-k,\phi_0(x)}$ and choose $\eta \in \Gamma(F^*_{-k})$ such that $\eta_{\phi_0(x)} = q^*$. The value $(\Phi\eta)_x$ is well-defined in $\odot E^*_x$ and has degree k. It suffices now to set

$$\phi'_r(p)(q^*) = \langle p, (\Phi\eta)_x \rangle \in \mathbb{R} \ , \tag{50}$$

where of course only the projection of $(\Phi\eta)_x$ onto $E^*_{-a_1,x} \odot \ldots \odot E^*_{-a_r,x}$ gives a nonzero contribution.

The definitions (48) and (50) are in fact inverses of each other. Indeed, if $p = X_{1,x} \odot \ldots \odot X_{r,x}$, $X_{j,x} \in E_{-a_j,x}$, choose $X_j \in \Gamma(E_{-a_j})$ (resp., $\eta_{k,1} \in \Gamma(F^*_{-k})$) that extends $X_{j,x}$ (resp., q^*). Definition (50) then reads

$$\phi'_r(X_{1,x} \odot \ldots \odot X_{r,x})(\eta_{k,1;\phi_0(x)}) = \langle X_{1,x} \odot \ldots \odot X_{r,x}, (\Phi^{k,r}\eta_{k,1})_x \rangle \ .$$

□

Corollary 1. *There is a 1:1 correspondence between graded vector bundle morphisms $\phi' : E \to F$ and bigraded algebra morphisms $\Phi : \Gamma(\odot F^*) \to \Gamma(\odot E^*)$, i.e. algebra morphisms that respect the standard and the homological degrees.*

Remark 12. We thus recover the result that the morphisms of split N-manifolds are the morphisms of graded vector bundles. Let us stress that the morphisms $\Phi : \Gamma(\odot F^*) \to \Gamma(\odot E^*)$ of graded algebras we considered in Proposition 3, are the morphisms of N-manifolds between the split N-manifolds $E[\cdot]$ and $F[\cdot]$ (split N-manifolds are not a full subcategory of N-manifolds).

Proof. The corollary is a direct consequence of the proof of the preceding proposition. Indeed, if ϕ'_1 is the unique map of the family of morphisms, it follows from Definition (48) that Φ respects both degrees. Conversely, if Φ is a bigraded algebra morphism, Equation (50) provides only a map ϕ'_1. □

The next theorem explains our definition of Lie n-algebroid morphisms.

Theorem 3. *There is a 1-to-1 correspondence between morphisms of split Lie n-algebroids from sE to sF and morphisms of differential graded algebras from $(\Gamma(\odot F^*), Q_F)$ to $(\Gamma(\odot E^*), Q_E)$.*

Remark 13. This theorem means that the morphisms between the split Lie n-algebroids (sE, ℓ_i, ρ) and (sF, m_i, r) are the morphisms of NQ-manifolds between the split NQ-manifolds $(E[\cdot], Q_E)$ and $(F[\cdot], Q_F)$. The point is that morphisms of split Lie n-algebroids are not necessarily morphisms of graded vector bundles. This observation is not surprising: morphisms of Lie infinity algebras are on their part usually not morphisms of graded vector spaces.

Let us first note that, in view of Proposition 3, Theorem 3 just means that the morphism conditions (35) and (36) are equivalent to the equivariance condition

$$Q_E \circ \Phi = \Phi \circ Q_F \tag{51}$$

– which proves that Definition 6 is independent of the chosen decompositions. More precisely, the equivariance condition is satisfied on the whole algebra $\Gamma(\odot F^*)$ if and only if it is satisfied on the generators $g \in C^\infty(N)$ and $\eta_{k,1} \in \Gamma(F^*_{-k})$, $k \in \{1, \ldots, n\}$. It will turn out that the condition (51) written on functions is equivalent to the condition (35), and that (51) written on generators of degrees $k \in \{1, \ldots, n\}$ is equivalent to the conditions (36).

Proof. To simplify, we work in this proof up to sign. However, some signs are needed to explain Definition 6. We denote them by $(\pm_1)-(\pm_3)$ and write them explicitly at the end of the proof.

Let $g \in C^\infty(N)$ and let $X \in \Gamma(E_{-1})$. We get

$$(Q_E^{1,1}\Phi^{0,0}g)(X) = \rho'(X)(\phi_0^* g)$$

and

$$(\Phi^{1,1}Q_F^{1,1}g)(X) = \langle (Q_F^{1,1}g)\circ\phi_0, \phi_1'\circ X\rangle = \sum_j f_j^X \langle Q_F^{1,1}g, \xi_j^X\rangle \circ \phi_0 = \sum_j f_j^X \phi_0^*(r'(\xi_j^X)g)\ .$$

In view of Equation (38), this means that $Q_E\circ\Phi$ and $\Phi\circ Q_F$ coincide on functions if and only if Equation (35) holds true.

Let now $k\in\{1,\ldots,n\}$, $\eta_{k,1}\in\Gamma(F_{-k}^*)$, and $t\in\{1,\ldots,k+1\}$. We will compute

$$\sum_{r=1}^{k+1} Q_E^{k+1,t}\Phi^{k,r}\eta_{k,1} - \sum_{r=1}^{k+1}\Phi^{k+1,t}Q_F^{k+1,r}\eta_{k,1} \in {}^t\mathscr{A}^{k+1} \tag{52}$$

on $(X_1,\ldots,X_t)$, $X_j\in\Gamma(E_{-a_j})$, $\sum a_j = k+1$.

When applying the definitions of Q_E and Φ, we get

$$\begin{aligned}
&\sum_{r=1}^{k+1}(Q_E^{k+1,t}\Phi^{k,r}\eta_{k,1})(X_1,\ldots,X_t)\\
&= \sum_{r=1}^{t}\sum_{\sigma\in \mathrm{Sh}(t-r+1,r-1)}(\Phi^{k,r}\eta_{k,1})(\ell'_{t-r+1}(X_{\sigma_1},\ldots,X_{\sigma_{t-r+1}})\odot X_{\sigma_{t-r+2}}\odot\ldots\odot X_{\sigma_t})\\
&\qquad +\sum_i \rho'(X_i)(\Phi^{k,t-1}\eta_{k,1})(X_1,\ldots\hat{\imath}\ldots,X_t)\\
&= \langle \eta_{k,1}\circ\phi_0, \sum_{r+s=t+1}\sum_{\sigma\in\mathrm{Sh}(s,r-1)}(\pm_1)\,\phi_r'\circ(\ell_s'(X_{\sigma_1},\ldots,X_{\sigma_s})\odot X_{\sigma_{s+1}}\odot\ldots\odot X_{\sigma_t})\rangle \qquad (53)\\
&\qquad +\sum_{i,j}\rho'(X_i)\left(f_j^{X_{\hat{\imath}}}\langle\eta_{k,1}\circ\phi_0,\xi_j^{X_{\hat{\imath}}}\circ\phi_0\rangle\right)\ ,
\end{aligned}$$

where $X_{\hat{\imath}}$ stands for $(X_1,\ldots\hat{\imath}\ldots,X_t)$. The last sum reads

$$\begin{aligned}
&\langle\eta_{k,1}\circ\phi_0,\sum_{i,j}(\pm_2)\,(\rho'(X_i)f_j^{X_{\hat{\imath}}})\xi_j^{X_{\hat{\imath}}}\circ\phi_0\rangle \qquad (54)\\
&\qquad +\sum_{i,j} f_j^{X_{\hat{\imath}}}\rho'(X_i)\phi_0^*\langle\eta_{k,1},\xi_j^{X_{\hat{\imath}}}\rangle\ .
\end{aligned}$$

Observe that, independently of the implication we have in mind, (35) and (36) imply (51) or (51) implies (35) and (36), we can assume at this stage that (35) and its equivalent form (38) hold true. It follows that the last sum of the preceding expression can be written in the form

$$\sum_{i,j,\ell} f_\ell^{X_i} f_j^{X_{\hat{\imath}}}\phi_0^*\left(r'(\xi_\ell^{X_i})\langle\eta_{k,1},\xi_j^{X_{\hat{\imath}}}\rangle\right)\ . \tag{55}$$

The reader has probably noticed that many of the terms we write and transform are zero. The point is that it is much easier to transform sums with potentially vanishing terms, than to work with the actually present terms.

On the other hand, when using Equation (34), we get

$$\sum_{r=1}^{k+1}(\Phi^{k+1,t}Q_F^{k+1,r}\eta_{k,1})(X_1,\ldots,X_t)$$

$$=\sum_{r=1}^{t}\frac{1}{r!}\sum_{\substack{t_1+\ldots+t_r=t\\ t_i\neq 0}}\sum_{\sigma\in\mathrm{Sh}(t_1,\ldots,t_r)}\sum_{j_1}\cdots\sum_{j_r}f_{j_1}^{X^{\sigma^1}}\ldots f_{j_r}^{X^{\sigma^r}}\,\phi_0^*\langle Q_F^{k+1,r}\eta_{k,1},\xi_{j_1}^{X^{\sigma^1}}\odot\ldots\odot\xi_{j_r}^{X^{\sigma^r}}\rangle$$

$$=\langle\eta_{k,1}\circ\phi_0,\sum_{r=1}^{t}\frac{1}{r!}\sum_{\substack{t_1+\ldots+t_r=t\\ t_i\neq 0}}\sum_{\sigma\in\mathrm{Sh}(t_1,\ldots,t_r)}\sum_{j_1}\cdots\sum_{j_r}(\pm_3)\,f_{j_1}^{X^{\sigma^1}}\ldots f_{j_r}^{X^{\sigma^r}}\,m'_r(\xi_{j_1}^{X^{\sigma^1}},\ldots,\xi_{j_r}^{X^{\sigma^r}})\circ\phi_0\rangle \tag{56}$$

$$+\frac{1}{2}\sum_{\substack{t_1+t_2=t\\ t_i\neq 0}}\sum_{\sigma\in\mathrm{Sh}(t_1,t_2)}\sum_{j_1}\sum_{j_2}f_{j_1}^{X^{\sigma^1}}f_{j_2}^{X^{\sigma^2}}\,\phi_0^*\left((r'\odot\eta_{k,1})(\xi_{j_1}^{X^{\sigma^1}},\xi_{j_2}^{X^{\sigma^2}})\right)\,.$$

We now examine the nonzero terms in the sum over t_1,t_2. Note first that if $t=1$, the entire sum vanishes. We thus can assume that $t\geq 2$. The function that we pull back by ϕ_0 is given by

$$r'(\xi_{j_1}^{X^{\sigma^1}})\langle\eta_{k,1},\xi_{j_2}^{X^{\sigma^2}}\rangle+r'(\xi_{j_2}^{X^{\sigma^2}})\langle\eta_{k,1},\xi_{j_1}^{X^{\sigma^1}}\rangle$$

and does therefore not vanish only if the sum of the degrees of $X_{\sigma_1},\ldots,X_{\sigma_{t_1}}$ or $X_{\sigma_{t_1+1}},\ldots,X_{\sigma_{t_1+t_2}}$ is -1. In this case, $t_1=1$ (and $t_2=t-1$) or $t_2=1$ (and $t_1=t-1$). As already observed above, the latter two possibilities correspond to different terms in the sum over t_1,t_2, only if $t\neq 2$. Moreover, since the sum of the degrees of $X_1,\ldots,X_t$ is $-k-1$, see above, we find that $t=2$, if $k=1$.

Consider first the case $t\neq 2$ (then $k\neq 1$). The sum over t_1,t_2 now reads

$$\frac{1}{2}2\sum_i\sum_{j_1}\sum_{j_2}f_{j_1}^{X_i}f_{j_2}^{X_{\hat{\imath}}}\phi_0^*\left(r'(\xi_{j_1}^{X_i})\langle\eta_{k,1},\xi_{j_2}^{X_{\hat{\imath}}}\rangle+r'(\xi_{j_2}^{X_{\hat{\imath}}})\langle\eta_{k,1},\xi_{j_1}^{X_i}\rangle\right)$$

$$=\sum_{i,j,\ell}f_\ell^{X_i}f_j^{X_{\hat{\imath}}}\phi_0^*\left(r'(\xi_\ell^{X_i})\langle\eta_{k,1},\xi_j^{X_{\hat{\imath}}}\rangle\right)\,, \tag{57}$$

since the degree of $\xi_{j_2}^{X_{\hat{\imath}}}$, i.e. the degree of $X_{\hat{\imath}}$, is less than -1.

In case $t=2$, the sum over t_1,t_2 reads

$$\frac{1}{2}2\sum_{j,\ell}f_\ell^{X_1}f_j^{X_2}\phi_0^*\left(r'(\xi_\ell^{X_1})\langle\eta_{k,1},\xi_j^{X_2}\rangle+r'(\xi_j^{X_2})\langle\eta_{k,1},\xi_\ell^{X_1}\rangle\right)$$

$$=\sum_{i,j,\ell}f_\ell^{X_i}f_j^{X_{\hat{\imath}}}\phi_0^*\left(r'(\xi_\ell^{X_i})\langle\eta_{k,1},\xi_j^{X_{\hat{\imath}}}\rangle\right)\,. \tag{58}$$

It now suffices to observe that the sum (55) and the sum (57) or (58) cancel out. Indeed, if the morphism condition (36) is satisfied, the difference (52) vanishes and $Q_E\circ\Phi-\Phi\circ Q_F$ vanishes on all generators. Conversely, if the difference (52) vanishes, the 'sum' of the evaluations (53), (54), and (56) vanishes at any point $x\in M$. As the second factor of each one of these evaluations is an element of $F_{-k,\phi_0(x)}$ and the first an arbitrary element $\eta_{k,1;\phi_0(x)}\in F^*_{-k,\phi_0(x)}$, this means that the condition (36) is verified. □

Addendum. It is straightforwardly checked that the signs $(\pm_1)-(\pm_3)$ are given by

$$(\pm_1)=(-1)^k\varepsilon(\sigma),(\pm_2)=(-1)^{\tilde{X}_i(\tilde{X}_1+\ldots+\tilde{X}_{i-1}+k)+1},\ \text{and}\ (\pm_3)=(-1)^k\varepsilon(\sigma)\,,$$

which completes the explanation of Definition 6.

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